EE320 Lecture Note
Chapter 4: Linear Model, Basic Matrix Algebra and Application

1 Terminology and Type of Matrix

$$\mathbb{A}_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m\times n}$$

Subscript of each a_{ij} indicates dimension of particular element in the matrix:

m × n = row × column

<u>Example</u> $\mathbb{B}_{2\times 2} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_{2\times 2} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$

$b_{11} = 1$
$b_{12} = 2$
$b_{21} = 3$
$b_{22} = 4$

Types

1. Square Matrix : no. of rows = no. of columns

$$\text{ex. } \mathbb{A}_{3\times 3} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

2. Identity Matrix: square matrix with diagonal elements being equal to 1 and the rest equal to 0

$$\text{ex. } \mathbb{I}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{\text{diagonal elements} = 1}$$

3. Column Vector: $\mathbb{A} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}_{1\times 3}$ when n=1

4. Row Vector: $\mathbb{B} = \begin{bmatrix} 4 \\ 3 \\ 6 \end{bmatrix}_{1\times 3}$ when m =1

2 Matrix Operation

2.1 Basic Operation

1. $\mathbb{A} = \mathbb{B}$ iff $a_{ij} = b_{ij} \quad \forall i \forall j$

2.

$$\left.\begin{aligned} \mathbb{A} =& \quad [a_{ij}]_{m\times n} \\ \mathbb{B} =& \quad [b_{ij}]_{m\times n} \end{aligned}\right\} \text{same direction}$$

$$\mathbb{A} \pm \mathbb{B} = [a_{ij}]_{m\times n} \pm [b_{ij}]_{m\times n} = [a_{ij} \pm b_{ij}]_{m\times n}$$

<u>ex.</u> $\mathbb{A} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ $\mathbb{A} + \mathbb{B} = \begin{bmatrix} 5 & 5 \\ 5 & 4 \end{bmatrix}$

$\mathbb{B} = \begin{bmatrix} 4 & 41 \\ 4 & 3 \end{bmatrix}$ $\mathbb{A} - \mathbb{B} = \begin{bmatrix} -3 & -3 \\ -3 & -2 \end{bmatrix}$

3. If α is a real number, then $\alpha\mathbb{A} = \alpha [a_{ij}]_{m\times n} = [\alpha a_{ij}]_{m\times n}$

<u>ex.</u> $5\mathbb{A} = 5 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 5 \\ 5 & 5 \end{bmatrix}$

4. Rules of matrix addition and multiplication by constants:

$$\begin{array}{lll} 1) & (\mathbb{A} + \mathbb{B}) + \mathbb{C} & = & \mathbb{A} + (\mathbb{B} + \mathbb{C}) \\ 2) & \mathbb{A} + \mathbb{B} & = & \mathbb{B} + \mathbb{A} \\ 3) & \mathbb{A} + \underline{0} & = & \mathbb{A} \\ 4) & \mathbb{A} + (-\mathbb{A}) & = & \underline{0} \\ 5) & (\alpha + \beta)\mathbb{A} & = & \alpha\mathbb{A} + \beta\mathbb{A} \\ 6) & \alpha(\mathbb{A} + \mathbb{B}) & = & \alpha\mathbb{A} + \alpha\mathbb{B} \end{array}$$

2.2 Matrix Multiplication

Suppose $\mathbb{A} = \left[a_{ij}\right]_{m\times \underline{n}}$ and $\mathbb{B} = \left[b_{ij}\right]_{\underline{n}\times p}$, then the product $\mathbb{C} = \mathbb{A}\mathbb{B} = \left[c_{ij}\right]_{m\times p}$

$c_{ij} = \sum_{r=1}^{n} a_{ir}b_{rj} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$

<u>ex.</u> $\mathbb{A}_{1\times 2} = \begin{bmatrix} a_{11} & a_{12} \end{bmatrix}$ $\mathbb{B}_{2\times 3} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}$

$$\mathbb{C}_{1\times 3} = \mathbb{A}_{1\times \underline{2}}\mathbb{B}_{\underline{2}\times 3} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} & a_{11}b_{13} + a_{12}b_{23} \end{bmatrix}_{1\times 3} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \end{bmatrix}$$

$\mathbb{D} = \mathbb{B}\cdot\mathbb{A}$ not possible since column $\neq$ row

<u>ex.</u> $\mathbb{A} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}_{2\times 1}$ $\mathbb{B} = \begin{bmatrix} 4 & 3 \end{bmatrix}_{1\times 2}$

$$\mathbb{C}_{2\times 2} = \mathbb{A}_{2\times 1}\cdot\mathbb{B}_{1\times 2} = \begin{bmatrix} 4 & 3 \\ 8 & 6 \end{bmatrix} \begin{bmatrix} 1\times 4 & 1\times 3 \\ 2\times 4 & 2\times 3 \end{bmatrix}$$

<u>ex.</u> $\mathbb{A} = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}_{2\times 2}$ $\mathbb{B} = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}_{2\times 2}$

$$\mathbb{C}_{2\times 2} = \mathbb{A}_{2\times 2}\cdot\mathbb{B}_{2\times 2} = \begin{bmatrix} (1\times 5) + (3\times 7) = c_{11} & (1\times 6) + (3\times 8) = c_{12} \\ (2\times 5) + (4\times 7) = c_{21} & (2\times 6) + (4\times 8) = c_{22} \end{bmatrix}_{2\times 2}$$

<u>ex.</u> $\mathbb{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $\mathbb{A}\mathbb{I}_2 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$\mathbb{I}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\mathbb{I}_2\mathbb{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$\therefore \mathbb{A}\mathbb{I} = \mathbb{I}\mathbb{A} = \mathbb{A}$ but $\mathbb{A}\mathbb{B} \neq \mathbb{B}\mathbb{A}$

Identity matrix is the matrix that when it is multiplied by the other matrix, the result is that other matrix multiplied.

<u>Rules for matrix multiplication</u>

1) $\mathbb{A}\mathbb{B} \neq \mathbb{B}\mathbb{A}$ in general

2) $(\mathbb{A}\mathbb{B})\mathbb{C} = \mathbb{A}(\mathbb{B}\mathbb{C})$

3) $(\mathbb{A} + \mathbb{B})\mathbb{C} = \mathbb{A}\mathbb{C} + \mathbb{B}\mathbb{C}$

2.3 Matrix Transposition

$$\mathbb{A} = \left[a_{ij}\right]_{m\times n}$$

$$\mathbb{A}' = \mathbb{A}^{\mathrm{T}} = \left[a_{ij}\right]_{n\times m}$$

ex. $\mathbb{A} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}_{2\times 1} \quad \rightarrow \mathbb{A}^{\mathrm{T}} = \begin{bmatrix} 1 & 2 \end{bmatrix}_{1\times 2}$

ex. $\mathbb{B} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_{2\times 2} \rightarrow \mathbb{B}^{\mathrm{T}} = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}_{2\times 2}$

Rules for transposes

1) $(\mathbb{A}^{\mathrm{T}})^{\mathrm{T}} = \mathbb{A}$
2) $(\mathbb{A}+\mathbb{B})^{\mathrm{T}} = \mathbb{A}^{\mathrm{T}} + \mathbb{B}^{\mathrm{T}}$
3) $(\mathbb{A}\mathbb{B})^{\mathrm{T}} = \mathbb{B}^{\mathrm{T}}\mathbb{A}^{\mathrm{T}}$
4) $(\alpha\mathbb{A})^{\mathrm{T}} = \alpha\mathbb{A}^{\mathrm{T}}$

ex. $\mathbb{A} = \begin{bmatrix} 3 & 1 \\ 4 & 7 \end{bmatrix} \quad \mathbb{B} = \begin{bmatrix} -5 & 2 \\ 6 & 0 \end{bmatrix}$

$$\mathbb{A}\mathbb{B} = \begin{bmatrix} -9 & 6 \\ 22 & 8 \end{bmatrix} \qquad (\mathbb{A}\mathbb{B})^{\mathrm{T}} = \begin{bmatrix} -9 & 22 \\ 6 & 8 \end{bmatrix}$$

$$\mathbb{B}^{\mathrm{T}}\mathbb{A}^{\mathrm{T}} = \begin{bmatrix} -5 & 6 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 1 & 7 \end{bmatrix} = \begin{bmatrix} -9 & 22 \\ 6 & 8 \end{bmatrix}$$

3 System of Linear Equation

$3x_1 + 4x_2 = 5$
$7x_1 + 4x_2 = 2$

Write this two equations and two unknowns in matrix form: $\mathbb{A}\mathbb{X} = b$

$$\overset{\mathbb{A}}{\begin{bmatrix} 3 & 4 \\ 7 & 4 \end{bmatrix}_{2\times\underline{2}}} \overset{\mathbb{X}}{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{\underline{2}\times 1}} = \overset{b}{\begin{bmatrix} 5 \\ 2 \end{bmatrix}}$$

$$\begin{bmatrix} 3x_1 + 4x_2 \\ 7x_1 + 4x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

So if we have m equations and n unknowns :

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \text{—①} \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \text{—②} \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \text{—ⓜ} \end{aligned}\right\} \overset{\mathbb{A}}{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}} \overset{\mathbb{X}}{\begin{bmatrix} x_1 \\ x_2 \\ \vdots x_n \end{bmatrix}} = \overset{b}{\begin{bmatrix} b_1 \\ b_2 \\ \vdots b_m \end{bmatrix}}$$

Consider general form of two equations with two unknowns:

$$\left.\begin{aligned} \text{①} \_\, a_{11}x_1 + a_{12}x_2 &= b_1 \\ \text{②} \_\, a_{21}x_1 + a_{22}x_2 &= b_2 \end{aligned}\right\} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \qquad , \mid A \mid = a_{11}a_{22} - a_{12}a_{21}$$

$$\begin{array}{rcrcl} \text{①} \times a_{21} & : & a_{11}a_{21}x_1 + a_{12}a_{21}x_2 & = & a_{21}b_1 \text{——③} \\ \text{②} \times a_{11} & : & a_{11}a_{21}x_1 + a_{11}a_{22}x_2 & = & a_{11}b_2 \text{——④} \\ \text{③} - \text{④} & : & (a_{12}a_{21} - a_{11}a_{22})x_2 & = & a_{21}b_1 - a_{11}b_2 \end{array}$$

$$x_2^* = \frac{a_{21}b_1 - a_{11}b_2}{a_{12}a_{21} - a_{11}a_{22}}$$

$$x_1^* = \frac{a_{12}b_2 - a_{22}b_1}{a_{12}a_{21} - a_{11}a_{22}}$$

<u>*ex.*</u>

$$\begin{array}{lrcl} & x + y & = & 10 \text{——①} \\ & 2x + y & = & 5 \text{ ——②} \\ \text{①} - \text{②} & -x & = & 5 \\ & x & = & 5 \\ & y & = & 15 \end{array}$$

or

$$\begin{aligned} \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 10 \\ 5 \end{bmatrix} \\ \mathbb{A}\mathbb{X} &= b \\ \mathbb{A}^{-1}\mathbb{A}\mathbb{X} &= \mathbb{A}^{-1}b \\ \mathbb{X} &= \mathbb{A}^{-1}b \end{aligned}$$

4 Matrix Inversion

$\mathbb{A}\mathbb{A}^{-1} = \mathbb{A}^{-1}\mathbb{A} = \mathbb{I}_\mathrm{n}$, $\mathbb{A}^{-\mathbb{1}}$ is defined iff $\mathbb{A}$ is square matrix

Properties

1) If $\mathbb{A}^{-1}$ exists, If $(\mathbb{A}^{-1})^{-1} = \mathbb{A}$

2) If $\mathbb{A}\mathbb{B}$ is invertible, then $(\mathbb{A}\mathbb{B})^{-1} = \mathbb{B}^{-1}\mathbb{A}^{-1}$

3) $(\mathbb{A}^{\mathbb{T}})^{-1} = (\mathbb{A}^{-1})^{\mathrm{T}}$

4) $(c\mathbb{A})^{-1} = (c^{-1}A^{-1})$ where c $\neq 0$

If a square matrix $\mathbb{A}$ has an inverse, $\mathbb{A}$ is said to be non singular.

If a square matrix $\mathbb{A}$ is not invertible, then $\mathbb{A}$ is called a singular matrix.

For system of linear equations: $\mathbb{A}\mathbb{X} = \mathrm{b}$

$$\mathbb{A}^{-1}\mathbb{A}\mathbb{X} = \mathbb{A}^{-1}\mathrm{b}$$

5 Determinant

To test whether matrix is invertible, we can use determinant.

Determinant of square matrix $\mathbb{A}$, $|\ A\ |$, is a uniquely defined number associated with that matrix.

$$|\ A_1\ | = |\ a\ | = a$$

$$|\ A_2\ | = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

$$|\ A_3\ | = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

① Laplace Expansion $\Rightarrow$ $a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$

② Sarrus's Law $\Rightarrow$ $\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{vmatrix}$

$$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12}$$

① Laplace Expansion

Let$\mathbb{A}$ be an n × n matrix.

- The "minor" of the element a_{ij} is $\mid \mathbb{M}_{\mathrm{ij}} \mid = \begin{vmatrix} a_{11} & \cdots & a_{1,j-1} & a_{1,j} & a_{1,j+1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{i1} & \cdots & a_{i,j-1} & a_{i,j} & a_{i,j+1} & \cdots & a_{i,n} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{n,j-1} & a_{n,j} & a_{n,j+1} & \cdots & a_{n,n} \end{vmatrix}$

- The "cofactor" of the element a_{ij} is $\mid C_{ij} \mid = (-1)^{i+j} \mid \mathbb{M}_{ij}$

$\mid |\mathbb{A}| = \sum_{j=1}^{n} a_{ij} \mid C_{ij} \mid$ expansion by the i^{th} row.

$= \sum_{i=1}^{n} a_{ij} \mid C_{ij} \mid$ expansion by the j^{th} column.

ex. $\mathbb{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$1^{st} row :\mid A \mid = a_{11} \mid c_{11} \mid + a_{12} \mid c_{12} \mid + a_{13} \mid c_{13} \mid$

$1^{st} column :\mid A \mid = a_{11} \mid c_{11} \mid + a_{21} \mid c_{21} \mid + a_{31} \mid c_{31} \mid$

$c_{11} = (-1)^{i+1} \mid \mathbb{M}_{11} \mid = (-1)^2 \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$

$c_{12} = (-1)^{i+2} \mid \mathbb{M}_{12} \mid = (-1)^3 \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$

Properties of Determinants

1) $\mid \mathbb{A} \mid = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4 - 6 = -2 \mid \mathbb{A}' \mid = \begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix} = 4 - 6 = -2$

2) The interchange of any two rows (or columns) will alter the sign, but not numerical value of determinant.

ex. $\mid \mathbb{A} \mid = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4 - 6 = -2$

$\mid \mathbb{B} \mid = \begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix} = 6 - 4 = -2$

$\mid \mathbb{A} \mid = \begin{vmatrix} 2 & 1 \\ 4 & 3 \end{vmatrix} = 6 - 4 = -2$

3) If all elements in any row (or column) is multiplied by a scalar k, the determinant is multiplied by k.

ex. $\mid \mathbb{A} \mid = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4 - 6 = -2$

$k = 2$

$\mid \mathbb{B} \mid = \begin{vmatrix} 2 & 4 \\ 3 & 4 \end{vmatrix} = 8 - 12 = -4$

$\mid \mathbb{C} \mid = \begin{vmatrix} 2 & 2 \\ 6 & 4 \end{vmatrix} = 8 - 12 = -4$

**** 4) The addition (or substitution) of a multiple of any row to (from) another row will leave the value of the determinant unaltered.

ex. $[\mathbb{A}] = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix}_{R_2 - 3R_1} \rightarrow \begin{vmatrix} 1 & 2 \\ 0 & -2 \end{vmatrix} = -2 - 0 = -2$

Not $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}_{2R_2 - 4R_1} \rightarrow \begin{vmatrix} 1 & 2 \\ 3 & 2 \end{vmatrix} = 0 - 4 = -4$

"Row Operation" *not column

ex. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \sim \begin{bmatrix} -1 & 2 \\ -1 & 4 \end{bmatrix}_{c_1 - c_2} \rightarrow \begin{vmatrix} -1 & 2 \\ -1 & 4 \end{vmatrix} = -4 - (-2) = -2$

5) If one row (or column) is a multiple of another row (or column), the value of the determinant will be zero.

ex. $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \rightarrow \begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = 4 - 4 = 0$

"linearly dependent" → one row/column depends on another.

Note The condition of linear independence is a sufficient condition for non singularity(invertible) of a matrix. For the rows (or column)to be linearly independent, none must be a linear combination of the real

*Nonsingularity : $\mid \mathbb{A} \mid \neq 0 \Leftrightarrow$ there is row(column) independence in $\mathbb{A}$ is singular $\mathbb{A}^{-1}$ exist a unique solution $\mathbb{X}^* = \mathbb{A}^{-1}d$ exists.

6 Matrix Inversion (Cont'd)

If $\mathbb{A}_{n\times n}$ is a nonsingular matrix and $\mid \mathbb{A} \mid \neq 0$

The inverse of $\mathbb{A}$ is $\mathbb{A}^{-1} = \frac{1}{|\mathbb{A}|}$ adj.$(\mathbb{A})$

$$\text{where} \quad \text{adj.}(\mathbb{A}) \equiv \mathbb{C}^T_{n\times n} \equiv \begin{bmatrix} \mid c_{11} \mid & \mid c_{21} \mid & \cdots & \mid c_{n1} \mid \\ \mid c_{12} \mid & \mid c_{22} \mid & \cdots & \mid c_{n2} \mid \\ \vdots & \vdots & \ddots & \vdots \\ \mid c_{1n} \mid & \mid c_{2n} \mid & \cdots & \mid c_{nn} \mid \end{bmatrix}$$

$$\text{and } \mid c_{ij} \mid = (-1)^{i+j} \begin{vmatrix} a_{11} & \cdots & a_{1,j-1} & a_{1,j} & a_{1,j+1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{i1} & \cdots & a_{i,j-1} & a_{i,j} & a_{i,j+1} & \cdots & a_{i,n} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{n,j-1} & a_{n,j} & a_{n,j+1} & \cdots & a_{n,n} \end{vmatrix}$$

ex. Find inverse of $\mathbb{A} = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & b \\ 0 & 1 & -d \end{bmatrix}$

Step I. Find $\mid \mathbb{A} \mid = \begin{vmatrix} 1 & -1 & 0 \\ 1 & 0 & b \\ 0 & 1 & -d \end{vmatrix}$ = 0+0+0-0-b-d = -b-d by Sarrus's Rule

Step II. Find cofactors of all elements of $\mathbb{A}$

$$\mid c_{11} \mid = (-1)^2 \begin{vmatrix} 0 & b \\ 1 & -d \end{vmatrix} = -b \qquad \mid c_{23} \mid = (-1)^5 \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = -1$$

$$\mid c_{12} \mid = (-1)^3 \begin{vmatrix} 1 & b \\ 0 & -d \end{vmatrix} = +d \qquad \mid c_{31} \mid = (-1)^4 \begin{vmatrix} -1 & 0 \\ 0 & b \end{vmatrix} = -b$$

$$\mid c_{13} \mid = (-1)^4 \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \qquad \mid c_{32} \mid = (-1)^5 \begin{vmatrix} 1 & 0 \\ 1 & b \end{vmatrix} = -b$$

$$\mid c_{21} \mid = (-1)^3 \begin{vmatrix} -1 & 0 \\ 1 & -d \end{vmatrix} = -d \qquad \mid c_{33} \mid = (-1)^6 \begin{vmatrix} 1 & -1 \\ 1 & 0 \end{vmatrix} = 1$$

$$\mid c_{22} \mid = (-1)^4 \begin{vmatrix} 1 & 0 \\ 0 & -d \end{vmatrix} = -d \qquad \mid c_{33} \mid = (-1)^6 \begin{vmatrix} 1 & -1 \\ 1 & 0 \end{vmatrix} = 1$$

Step III. Take transpose

$$\therefore adj.(\mathbb{A}) = C^T_{n\times n} = \begin{bmatrix} -b & -d & -b \\ +d & -d & -b \\ 1 & -1 & 1 \end{bmatrix}$$

Step IV. $\mathbb{A}^{-1} = \frac{1}{|\mathbb{A}|} adj.(\mathbb{A}) = \frac{1}{-(b+d)} \begin{bmatrix} -b & -d & -b \\ +d & -d & -b \\ 1 & -1 & 1 \end{bmatrix}$

<u>Ex.</u>

$$\begin{aligned} Q^d &= Q^s \\ Q^d &= a - bP \\ Q^s &= -c + dP \end{aligned}$$

$$\Rightarrow \quad \begin{aligned} Q^d - Q^s + 0P &= 0 \\ Q^d + 0Q^s + bP &= a \\ Q^d + Q^s - dP &= -c \end{aligned}$$

$$\Rightarrow \quad \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & b \\ 0 & 1 & -d \end{bmatrix} \begin{bmatrix} Q^d \\ Q^s \\ P \end{bmatrix} = \begin{bmatrix} 0 \\ a \\ -c \end{bmatrix}$$

$$\begin{aligned} \Rightarrow \quad \mathbb{A}_{3\times3}\mathbb{X}_{3\times1} &= b_{3\times1} \\ \mathbb{A}^{-1}_{3\times3}\mathbb{A}_{3\times3}\mathbb{X}_{3\times1} &= \mathbb{A}^{-1}_{3\times3}b_{3\times1} \\ \mathbb{I}_{3\times3}\mathbb{X}_{3\times1} &= \mathbb{A}^{-1}_{3\times3}b_{3\times1} \\ \mathbb{X}^*_{3\times1} &= \mathbb{A}^{-1}_{3\times3}b_{3\times1} \end{aligned}$$

$$\begin{aligned} \therefore \mathbb{X}^* = \begin{bmatrix} Q^{d*} \\ Q^{s*} \\ P^* \end{bmatrix} = \mathbb{A}^{-1}b &= \frac{1}{-(b-d)} \begin{bmatrix} -b & -d & -b \\ +d & -d & -b \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ a \\ -c \end{bmatrix} \\ &= \frac{1}{-(b-d)} \begin{bmatrix} bc - ad \\ bc - ad \\ -a - c \end{bmatrix} \\ \mathbb{X}^* &= \begin{bmatrix} \frac{ad-bc}{b+d} \\ \frac{ad-bc}{b+d} \\ \frac{a+c}{b+d} \end{bmatrix} = \begin{bmatrix} Q^{d*} \\ Q^{s*} \\ P^* \end{bmatrix} \end{aligned}$$

7 Cramer's Rule

Given an equation system $\mathbb{A}\mathbb{X} = \mathrm{b}$, where $\mathbb{A}$ is n × n nonsingular matrix, the solution value of j^{th} variable can be obtained from

$$\mathbb{X}_j^* = \frac{|\mathbb{A}_j|}{|\mathbb{A}|} \begin{vmatrix} a_{11} & a_{12} & \cdots & d_1 & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & d_2 & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & d_n & \cdots & a_{nn} \end{vmatrix}$$

where $|\ \mathbb{A}_j\ |$ is the determinant of $\mathbb{A}$ when the j^{th} column is replaced by the constant terms $d_1\ ,\ \cdots, d_n$

<u>ex.</u> $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix}$

$$\text{By Cramer's Rule: } x^* = \frac{\begin{vmatrix} u & b \\ v & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} = \frac{|\mathbb{A}_1|}{|\mathbb{A}|} = \frac{ud-bv}{ad-bc}$$

$$y^* = \frac{\begin{vmatrix} a & u \\ c & v \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} = \frac{|\mathbb{A}_2|}{|\mathbb{A}|} = \frac{av-uc}{ad-bc}$$

ex. $$\begin{bmatrix}1 & -1 & 0\\ 1 & 0 & b\\ 0 & 1 & -d\end{bmatrix}\begin{bmatrix}Q^d\\ Q^s\\ P\end{bmatrix}=\begin{bmatrix}0\\ a\\ -c\end{bmatrix}$$

$$Q^{d*} = \frac{\begin{vmatrix}0 & -1 & 0\\ a & 0 & b\\ -c & 1 & -d\end{vmatrix}}{\begin{vmatrix}1 & -1 & 0\\ 1 & 0 & b\\ 0 & 1 & -d\end{vmatrix}} = \frac{a_{11}|c_{11}|+a_{12}|c_{12}|+a_{13}|c_{13}|}{-(b+d)} = \frac{(-1)(-1)^{2+1}\begin{vmatrix}a & b\\ -c & -d\end{vmatrix}}{-(b+d)} = \frac{ad-bc}{b+d}$$

$$Q^{s*} = \frac{\begin{vmatrix}1 & 0 & 0\\ 1 & a & b\\ 0 & -c & -d\end{vmatrix}}{\begin{vmatrix}1 & -1 & 0\\ 1 & 0 & b\\ 0 & 1 & -d\end{vmatrix}} = \frac{a_{11}|c_{11}|+a_{12}|c_{12}|+a_{13}|c_{13}|}{-(b+d)} = \frac{(-1)(-1)^{2}\begin{vmatrix}a & b\\ -c & -d\end{vmatrix}}{-(b+d)} = \frac{ad-bc}{b+d}$$

$$P^{*} = \frac{\begin{vmatrix}1 & -1 & 0\\ 1 & 0 & a\\ 0 & 1 & -c\end{vmatrix}}{\begin{vmatrix}1 & -1 & 0\\ 1 & 0 & b\\ 0 & 1 & -d\end{vmatrix}} = \frac{a_{11}|c_{11}|+a_{12}|c_{12}|+a_{13}|c_{13}|}{-(b+d)}$$

$$= \frac{1(-1)^{2}\begin{vmatrix}0 & a\\ 1 & -c\end{vmatrix}+(-1)(-1)^{3}\begin{vmatrix}1 & a\\ 0 & -c\end{vmatrix}}{-(b+d)} = \frac{-a-c}{-(b+d)} = \frac{a+c}{b+d} \quad \#$$

8 Matrix Algebra : Application

✓ Specific tax on producer

$$\begin{aligned} Q^d &= Q^s \\ Q^d &= a - bP \\ Q^s &= -c + d(P - t) \end{aligned}$$

$$\Rightarrow \qquad \begin{aligned} Q^d - Q^s + 0P &= 0 \\ Q^d + 0Q^s + bP &= a \\ Q^d + Q^s - dP &= -c - dt \end{aligned}$$

$$\Rightarrow \qquad \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & b \\ 0 & 1 & -d \end{bmatrix} \begin{bmatrix} Q^d \\ Q^s \\ P \end{bmatrix} = \begin{bmatrix} 0 \\ a \\ -c - dt \end{bmatrix}$$

$\Rightarrow$ Solve for Q^* and P^* using both matrix inversion and Cramer's rule.

$$Q^* = \frac{\begin{vmatrix} 0 & -1 & 0 \\ a & 0 & b \\ -c - dt & 1 & -d \end{vmatrix}}{\begin{vmatrix} 1 & -1 & 0 \\ 1 & 0 & b \\ 0 & 1 & -d \end{vmatrix}} = \frac{(-1)(-1)^3\begin{vmatrix} a & b \\ -c - dt & -d \end{vmatrix}}{-(b+d)} = \frac{-ad+b(c+dt)}{-(b+d)} = \frac{ad-bc-bdt}{b+d}$$

$$P^* = \frac{\begin{vmatrix} 1 & -1 & 0 \\ 1 & 0 & a \\ 0 & 1 & -c - dt \end{vmatrix}}{\begin{vmatrix} 1 & -1 & 0 \\ 1 & 0 & b \\ 0 & 1 & -d \end{vmatrix}} = \frac{1(-1)^2\begin{vmatrix} 0 & a \\ 1 & -c - dt \end{vmatrix} + (-1)(-1)^3\begin{vmatrix} 1 & a \\ 0 & -c - dt \end{vmatrix}}{-(b+d)}$$

$$= \frac{-a+(-c-dt)}{-(b+d)} = \frac{a+c+dt}{b+d}$$

✓ IS-LM model

$$\begin{aligned} C &= C_0 + cY_D \qquad & M^d = L_0 + l_1Y - l_2r \\ I &= I_0 - jr & M^s = M_0 \\ G &= G_0 \\ Y_D &= Y \\ Y &= C + I + G \end{aligned}$$

<u>Solve for IS-LM equilibrium</u>

@equilibrium in goods market :

$$\begin{aligned} Y &= C + I + G \\ Y &= C_0 + cY_D + I_0 - jr + G_0 \\ (1-c)Y + jr &= C_0 + I_0 + G_0 \end{aligned}$$

@equilibrium in money market :

$$\begin{aligned} M^d &= M^s \\ L_0 + l_1Y - l_2r &= M_0 \\ l_1Y - l_2r &= M_0 - L_0 \end{aligned}$$

$$\begin{bmatrix} 1-c & j \\ l_1 & -l_2 \end{bmatrix} \begin{bmatrix} Y \\ r \end{bmatrix} = \begin{bmatrix} C_0 + I_0 + G_0 \\ M_0 - L_0 \end{bmatrix}$$

$$Y^* = \frac{\begin{vmatrix} C_0 + I_0 + G_0 & j \\ M_0 - L_0 & -l_2 \end{vmatrix}}{\begin{vmatrix} 1-c & j \\ l_1 & -l_2 \end{vmatrix}} = \frac{-l_2(C_0+I_0+G_0)-j(M_0-L_0)}{-l_2(1-c)-l_1j} = \frac{l_2(C_0+I_0+G_0)+j(M_0-L_0)}{l_2(1-c)+l_1j}$$

$$r^* = \frac{\begin{vmatrix} 1-c & C_0 + I_0 + G_0 \\ l_1 & M_0 - L_0 \end{vmatrix}}{\begin{vmatrix} 1-c & j \\ l_1 & -l_2 \end{vmatrix}} = \frac{(1-c)(M_0-L_0)-l_1(C_0+I_0+G_0)}{-l_2(1-c)-l_1j} = \frac{l_1(C_0+I_0+G_0)-(1-c)(M_0-L_0}{l_2(1-c)+l_1j}$$

<u>ex.</u> Solve for endogeneous variables Y, r , I, C simultaneously.

$$\left.\begin{aligned} Y &= C + I + G_0 \\ C &= C_0 + cY \\ I &= I_0 - jr \\ M_0 &= L_0 + l_1Y - l_2r \end{aligned}\right\} \quad \begin{aligned} Y - C - I + 0r &= G_0 \\ -cY + C + 0I + 0r &= C_0 \\ 0Y + 0C + I + jr &= I_0 \\ l_1Y + 0C + 0I - l_2r &= M_0 - L_0 \end{aligned}$$

$$\Rightarrow \begin{bmatrix} 1 & -1 & -1 & 0 \\ -c & 1 & 0 & 0 \\ 0 & 0 & 1 & j \\ l_1 & 0 & 0 & -l_2 \end{bmatrix} \begin{bmatrix} Y \\ C \\ I \\ r \end{bmatrix} = \begin{bmatrix} G_0 \\ C_0 \\ I_0 \\ M_0 - L_0 \end{bmatrix}$$