Example 1.2. (Sum of a Geometric Sequence)
Use mathematical induction to prove that for any real number r except 1, and any integer $n \geq 0$,

$$P(n): \sum_{i=0}^{n} r^i = \frac{r^{n+1} - 1}{r - 1}.$$

$n=3$

$$\underbrace{r^0 + r^1 + r^2 + r^3}_{\sum_{i=0}^{3} r^i} = \frac{r^{3+1} - 1}{r-1}$$

$i=0 \quad i=1 \quad i=2 \quad i=3$

Induction Proof: $P(n)$, $\forall n \in \mathbb{Z}$, $n \geq 0$

(I) Basis step: show $P(0)$ is true.

$$P(0): \underbrace{\sum_{i=0}^{0} r^i}_{} = \frac{r^{0+1} - 1}{r - 1}$$

$$\underbrace{r^0}_{=1} = \underbrace{\frac{\cancel{r-1}}{\cancel{r-1}}}_{=1} \Rightarrow P(0) \text{ is true.}$$

(II) Inductive Step: $P(k) \to P(k+1)$, $\forall k \in \mathbb{Z}$, $k \geq 0$

Assume $P(k)$: $\boxed{\sum_{i=0}^{k} r^i = \frac{r^{k+1} - 1}{r - 1}}$ is true. — (*) ✓ Inductive hypothesis

Show: $P(k+1)$: $\sum_{i=0}^{k+1} r^i = \frac{r^{(k+1)+1} - 1}{r - 1}$ is true.

$$\sum_{i=0}^{k+1} r^i = \underbrace{r^0 + r^1 + \dots + r^k}_{(*)} + r^{k+1} = \underbrace{\sum_{i=0}^{k} r^i}_{(*)} + r^{k+1}$$

$$= \frac{(r^{k+1} - 1)}{r - 1} + r^{k+1}$$

$$= \frac{(r^{k+1} - 1) + (r-1) r^{k+1}}{r - 1}$$

$$= \frac{(\cancel{r^{k+1}} - 1) + (r^{(k+1)+1} - \cancel{r^{k+1}})}{r - 1}$$

$$= \frac{r^{(k+1)+1} - 1}{r - 1}$$

$\therefore$ $P(k+1)$ is true

Or, $P(k) \to P(k+1)$

From (I) and (II), $P(n)$ is true $\forall n \in \mathbb{Z}$, $n \geq 0$