Answers to Chapter 5 Exercises

1. **Consider the following consumption and portfolio choice problem. Assume that $U(C_t, t) = \delta^t[aC_t - bC_t^2]$, $B(W_T, T) = 0$, and $y_t \neq 0$, where $\delta = \frac{1}{1+\rho}$ and $\rho \geq 0$ is the individual's subjective rate of time preference. Further, assume that $n = 0$ so that there are no risky assets but there is a single-period riskless asset yielding a return of $R_{ft} = 1/\delta$ that is constant each period (equivalently, the risk-free interest rate $r_f = \rho$). Note that in this problem labor income is stochastic and there is only one (riskless) asset for the individual consumer-investor to hold. Hence, the individual has no portfolio choice decision but must decide only what to consume each period. In solving this problem, assume that the individual's optimal level of consumption remains below the "bliss point" of the quadratic utility function, that is, $C_t^* < \frac{1}{2} a/b, \forall t$.**

   a. **Write down the individual's wealth accumulation equation from period t to period $t+1$.**

   Answer:

$$W_{t+1} = R_{ft}[W_t + y_t - C_t] = \frac{1}{\delta}[W_t + y_t - C_t] \tag{1}$$

   b. **Solve for the individual's optimal level of consumption at date $T-1$ and evaluate $J(W_{T-1}, T-1)$.** *Hint:* **this is trivial.**

   Answer: **We assume throughout that the individual is below the maximal "bliss point" of the quadratic utility function, that is, $C^* < \frac{1}{2} a/b$. The solution at date $T-1$ is trivial. Since there is no bequest function, in the last period the individual consumes income and remaining wealth, $C^* = W_{T-1} + y_{T-1}$. Since we define**

$$\begin{aligned} J(W_t, t) &\equiv \max E_t\left[\sum_{s=t}^{T-1} U(C_s, s) + B(W_T, T)\right] \\ &= \max E_t\left[\sum_{s=t}^{T-1} \delta^s\left[aC_s - bC_s^2\right]\right] \end{aligned} \tag{2}$$

   Thus,

$$\begin{aligned} J(W_{T-1}, T-1) &= \delta^{T-1}\left[aC_{T-1}^* - bC_{T-1}^{*2}\right] \\ &= \delta^{T-1}[a(W_{T-1} + y_{T-1}) - b(W_{T-1} + y_{T-1})^2] \end{aligned} \tag{3}$$

   c. **Continue to solve the individual's problem at date $T-2$, $T-3$, and so on—and notice the pattern that emerges. From these results, solve for the individual's optimal level of consumption for any arbitrary date, $T-t$, in terms of the individual's expected future levels of income.**

   Answer: **Using**

$$J(W_t, t) = \max_{C_t}[U(C_t, t) + E_t[J(W_{t+1}, t+1)]] \tag{4}$$

at date $T-2$ we have

$$J(W_{T-2}, T-2) = \max_{C_{T-2}} \delta^{T-2}\left[aC_{T-2} - bC_{T-2}\right]^2 + E_{T-2}[\delta^{T-1}[a(W_{T-1} + y_{T-1}) - b(W_{T-1} + y_{T-1})^2]] \tag{5}$$

Substituting in $W_{T-1} = \frac{1}{\delta}[W_{T-2} + y_{T-2} - C_{T-2}]$ in (5) and taking the first order condition with respect to C_{T-2}, we have

$$\delta^{T-2}[a - 2bC_{T-2}] + \delta^{T-1}\left[-\frac{1}{\delta}a + \frac{2b}{\delta}\left[\frac{1}{\delta}(W_{T-2} + y_{T-2} - C_{T-2}) + E_{T-2}[y_{T-1}]\right]\right] = 0 \tag{6}$$

Thus,

$$C^*_{T-2} = \frac{1}{1+\delta}[W_{T-2} + y_{T-2} + \delta E_{T-2}[y_{T-1}]] \tag{7}$$

and therefore from (5)

$$\begin{aligned} J(W_{T-2}, T-2) &= \delta^{T-2}\left[aC^*_{T-2} - bC^{*\,2}_{T-2}\right] \\ &+ \delta^{T-1}E_{T-2}\left[a\left(\frac{1}{\delta}\left[W_{T-2} + y_{T-2} - C^*_{T-2}\right] + y_{T-1}\right) - b\left(\frac{1}{\delta}\left[W_{T-2} + y_{T-2} - C^*_{T-2}\right] + y_{T-1}\right)^2\right] \\ &= \delta^{T-2}\left\{\frac{a}{1+\delta}\left[W_{T-2} + y_{T-2} + \delta E_{T-2}[y_{T-1}]\right] - \frac{b}{(1+\delta)^2}\left[W_{T-2} + y_{T-2} + \delta E_{T-2}\left[y_{T-1}\right]\right]^2\right\} \\ &+ \delta^{T-1}E_{T-2}\left[\frac{a}{1+\delta}(W_{T-2} + y_{T-2} + \delta y_{T-1}) - \frac{b}{\delta(1+\delta)^2}(W_{T-2} + y_{T-2} + \delta y_{T-1})^2\right] \\ &= \delta^{T-2}\left[a[W_{T-2} + y_{T-2} + \delta E_{T-2}[y_{T-1}]] - \frac{b}{\delta(1+\delta)}\left[W_{T-2} + y_{T-2} + \delta E_{T-2}[y_{T-1}]\right]^2\right] \end{aligned} \tag{8}$$

Next, considering the problem at $T-3$, we have

$$J(W_{T-3}, T-3) = \max_{C_{T-3}} \delta^{T-3}\left[aC_{T-3} - bC_{T-3}\right]^2 + E_{T-3}[J(W_{T-2}, T-2)] \tag{9}$$

Substituting (8) into (9) and then substituting (7) for $W_{T-2} = \frac{1}{\delta}[W_{T-2} + y_{T-3} - C_{T-3}]$, we can then take the first order condition with respect to C_{T-3}. This leads to

$$C^*_{T-3} = \frac{1}{1+\delta+\delta^2}(W_{T-3} + y_{T-3} + E_{T-3}[\delta y_{T-2} + \delta^2 y_{T-1}]) \tag{10}$$

By now, we can see the pattern for the optimal consumption rule:

$$C^*_{T-m} = \frac{1}{1+\delta+L+\delta^{m-1}}\left(W_{T-m} + E_{T-m}\left[\sum_{t=0}^{m-1}\delta^t y_{T-m+t}\right]\right) \tag{11}$$

The rule is to consume a proportion of current "permanent income" where m is the number of years remaining in one's lifetime.

2. Consider the consumption and portfolio choice problem with power utility $U(C_t, t) \equiv \delta^t C_t^\gamma/\gamma$ and a power bequest function $B(W_T, T) \equiv \delta^T W_T^\gamma/\gamma$. Assume there is no wage income ($y_t \equiv 0 \ \forall \ t$) and a constant risk-free return equal to $R_{ft} = R_f$. Also, assume that $n = 1$ and the return of the single risky asset, R_{rt}, is independently and identically distributed over time. Denote the proportion of wealth invested in the risky asset at date t as ω_t.

   a. Derive the first-order conditions for the optimal consumption level and portfolio weight at date $T-1$, C_{T-1}^* and ω_{T-1}^*, and give an explicit expression for C_{T-1}^*.

Answer: Letting $S_t \equiv W_t - C_t$ and $R_t \equiv R_f + \omega_t^*(R_{rt} - R_f)$

$$\begin{aligned} U_C &= E_{T-1}[B_W R_{T-1}] \\ \delta^{T-1} C_{T-1}^{\gamma-1} &= E_{T-1}\left[\delta^T W_T^{\gamma-1} R_{T-1}\right] = E_{T-1}[\delta^T (S_{T-1} R_{T-1})^{\gamma-1} R_{T-1}] \\ &= \delta^T E_{T-1}\left[R_{T-1}^\gamma\right](W_{T-1} - C_{T-1})^{\gamma-1} \end{aligned} \tag{1}$$

or

$$\begin{aligned} C_{T-1} &= \frac{\left(\delta E_{T-1}\left[R_{T-1}^\gamma\right]\right)^{\frac{1}{\gamma-1}}}{1+\left(\delta E_{T-1}\left[R_{T-1}^\gamma\right]\right)^{\frac{1}{\gamma-1}}} W_{T-1} \\ &= \frac{a_1}{1+a_1} W_{T-1} = c_1 W_{T-1} \end{aligned} \tag{2}$$

where $c_1 = a_1/(1+a_1)$ and $a_1 \equiv \left(\delta E_{T-1}\left[R_{T-1}^\gamma\right]\right)^{\frac{1}{\gamma-1}} = \left(\delta E\left[R_{T-1}^\gamma\right]\right)^{\frac{1}{\gamma-1}}$ where $E[\cdot]$ is the unconditional expectations operator. The unconditional expectation is the same as the conditional one because the distribution of asset returns is assumed to be independent and identically distributed. The first order condition with respect to the portfolio weight implies

$$\begin{aligned} E_{T-1}[B_W R_{r,T-1}] &= R_f E_{T-1}[B_W] \\ \delta^T E_{T-1}\left[\left(S_{T-1} R_{T-1}\right)^{\gamma-1} R_{r,T-1}\right] &= \delta^T R_{f,T-1} E_{T-1}\left[(S_{T-1} R_{T-1})^{\gamma-1}\right] \\ E_{T-1}\left[R_{T-1}^{\gamma-1} R_{r,T-1}\right] &= R_{f,T-1} E_{T-1}\left[R_{T-1}^{\gamma-1}\right] \\ E\left[R_{T-1}^{\gamma-1} R_{r,T-1}\right] &= R_{f,T-1} E\left[R_{T-1}^{\gamma-1}\right] \end{aligned} \tag{3}$$

Equation (3) shows that the optimal portfolio weight is determined independently from the level of wealth or consumption. It depends only on the distribution of returns on the risky asset. Since this distribution is independent and identically distributed, we can again replace the conditional with the unconditional expectations operator. Hence, all individuals with the same constant coefficient of relative risk aversion, γ, chose the same portfolio proportions for the risky and risk-free asset.

b. Solve for the form of $J(W_{T-1}, T-1)$.

Answer:

$$\begin{aligned} J(W_{T-1}, T-1) &= \delta^{T-1} C_{T-1}^{*\gamma}/\gamma + \delta^{T} E_{T-1}\left[\left(R_{T-1}^{*}\left(W_{T-1} - C_{T-1}^{*}\right)\right)^{\gamma}/\gamma\right] \\ &= \delta^{T-1}\left(\frac{a_1}{1+a_1}\right)^{\gamma} W_{T-1}^{\gamma}/\gamma + \delta^{T} E_{T-1}\left[R_{T-1}^{*\gamma}\frac{W_{T-1}^{\gamma}}{\gamma(1+a_1)^{\gamma}}\right] \\ &= \delta^{T-1}\frac{W_{T-1}^{\gamma}}{\gamma(1+a_1)^{\gamma}}\left(a_1^{\gamma} + \delta E_{T-1}\left[R_{T-1}^{*\gamma}\right]\right) \\ &= \delta^{T-1} b_1 W_{T-1}^{\gamma}/\gamma \end{aligned} \tag{4}$$

where $b_1 \equiv \left(a_1^{\gamma} + \delta E\left[R_{T-1}^{*\gamma}\right]\right)/\left(1+a_1\right)^{\gamma} = \left[a_1/\left(1+a_1\right)\right]^{\gamma-1}$.

c. Derive the first-order conditions for the optimal consumption level and portfolio weight at date $T-2$, C_{T-2}^{*} and ω_{T-2}^{*}, and give an explicit expression for C_{T-2}^{*}.

Answer: **The optimality condition for consumption is**

$$\begin{aligned} U_C(C_{T-2}^{*}, T-2) &= E_{T-2}[J_W(W_{T-1}, T-1) R_{T-2}] \\ C_{T-2}^{\gamma-1} &= \delta^{T-1} E_{T-2}\left[b_1 W_{T-1}^{\gamma-1} R_{T-2}\right] \\ C_{T-2}^{\gamma-1} &= \delta E_{T-2}\left[b_1\left(S_{T-2}R_{T-2}\right)^{\gamma-1} R_{T-2}\right] \\ &= \delta b_1 E_{T-2}\left[R_{T-2}^{\gamma}\right]\left(W_{T-2} - C_{T-2}\right)^{\gamma-1} \end{aligned} \tag{5}$$

or

$$\begin{aligned} C_{T-2}^{*} &= \frac{\left(\delta b_1 E_{T-2}\left[R_{T-2}^{\gamma}\right]\right)^{\frac{1}{\gamma-1}}}{1 + \left(\delta b_1 E_{T-1}\left[R_{T-2}^{\gamma}\right]\right)^{\frac{1}{\gamma-1}}} W_{T-2} \\ &= \frac{a_2}{1+a_2} W_{T-2} \\ &= c_2 W_{T-2} \end{aligned} \tag{6}$$

where $c_2 = a_2/(1+a_2)$ and $a_2 \equiv \left(b_1 \delta E\left[R_{T-2}^{\gamma}\right]\right)^{\frac{1}{\gamma-1}} = a_1^2/(1+a_1) = a_1 c_1$. The optimality condition for the portfolio weight ω_{T-2}^{*} turns out to be of the same form as at $T-1$:

$$\begin{aligned} E_{T-2}[J_W R_{r,T-2}] &= R_f E_{T-2}[J_W] \\ E_{T-2}[b_1(S_{T-2}R_{T-2})^{\gamma-1} R_{r,T-2}] &= R_f E_{T-1}[b_1(S_{T-2}R_{T-2})^{\gamma-1}] \\ E_{T-2}\left[b_1 R_{T-2}^{\gamma-1} R_{r,T-2}\right] &= R_f E_{T-2}\left[b_1 R_{T-2}^{\gamma-1}\right] \\ E\left[R_{T-2}^{\gamma-1} R_{r,T-2}\right] &= R_f E\left[R_{T-2}^{\gamma-1}\right] \end{aligned} \tag{7}$$

Hence the portfolio weight ω_{T-2}^{*} is the same as ω_{T-1}^{*} and does not depend on wealth or consumption, just the distribution of risky asset returns.

d. Solve for the form of $J(W_{T-2}, T-2)$. Based on the pattern for $T-1$ and $T-2$, provide expressions for the optimal consumption and portfolio weight at any date $T-t$, $t=1, 2, 3, \ldots$.

Answer:

$$\begin{aligned}
J(W_{T-2}, T-2) &= U(C^*_{T-2}, T-2) + E_{T-1}[J(W_{T-1}, T-1)] \\
&= \delta^{T-2} C^{*\gamma}_{T-2}/\gamma + E_{T-1}\left[\delta^{T-1} b_1 W^{\gamma}_{T-1}/\gamma\right] \\
&= \delta^{T-2}\left(\frac{a_2}{1+a_2}\right)^{\gamma} W^{\gamma}_{T-2}/\gamma + \delta^{T-1} E_{T-1}\left[b_1\left(R^*_{T-2}\left(W_{T-2} - C^*_{T-2}\right)\right)^{\gamma}/\gamma\right] \\
&= \delta^{T-2}\left(\frac{a_2}{1+a_2}\right)^{\gamma} W^{\gamma}_{T-2}/\gamma + \delta^{T-1} E_{T-1}\left[b_1 R^{*\gamma}_{T-2} \frac{W^{\gamma}_{T-2}}{\gamma(1+a_2)^{\gamma}}\right] \\
&= \delta^{T-2} \frac{W^{\gamma}_{T-2}}{\gamma(1+a_2)^{\gamma}}\left(a_2^{\gamma} + \delta E_{T-2}\left[b_1 R^{*\gamma}_{T-2}\right]\right) \\
&= \delta^{T-1} b_2 W^{\gamma}_{T-2}/\gamma
\end{aligned} \tag{8}$$

where $b_2 \equiv \left(a_2^{\gamma} + \delta E\left[R^{*\gamma}_{T-2}\right]\right)/(1+a_2)^{\gamma}$.

The optimal portfolio weight is the same each period and satisfies

$$E\left[R^{\gamma-1}_{T-2} R_{r,T-2}\right] = R_f E\left[R^{\gamma-1}_{T-2}\right] \tag{9}$$

The level of consumption at $T-1$ is $C^*_{T-2} = c_2 W_{T-2}$ where $c_2 = a_1 c_1/(1+a_1 c_1)$. Hence, $C^*_{T-3} = c_3 W_{T-3}$ where $c_3 = a_1 c_2/(1+a_1 c_2)$. In general, $C^*_{T-t} = c_t W_{T-t}$ where $c_t = a_1 c_{t-1}/(1+a_1 c_{t-1})$.

3. Consider the multiperiod consumption and portfolio choice problem

$$\max_{C_s, \omega_s \forall s} E_t\left[\sum_{s=t}^{T-1} U(C_s, s) + B(W_T, T)\right]$$

Assume negative exponential utility $U(C_s, s) \equiv -\delta^s e^{-bC_s}$ and a bequest function $B(W_T, T) \equiv -\delta^T e^{-bW_T}$ where $\delta = e^{-\rho}$ and $\rho > 0$ is the (continuously compounded) rate of time preference. Assume there is no wage income ($y_s \equiv 0 \ \forall \ s$) and a constant risk-free return equal to $R_{fs} = R_f$. Also, assume that $n=1$ and the return of the single risky asset, R_{rs}, has an identical and independent normal distribution of $N(\overline{R}, \sigma^2)$ each period. Denote the proportion of wealth invested in the risky asset at date s as ω_s.

a. Derive the optimal portfolio weight at date $T-1$, ω^*_{T-1}. Hint: it might be easiest to evaluate expectations in the objective function prior to taking the first-order condition.

Answer: **The individual's problem at date $T-1$ is** $R_t \equiv R_f + \omega^*_t(R_{rt} - R_f)$

$$\begin{aligned}
&\max_{C_{T-1},\omega_{T-1}} -\delta^{T-1}e^{-bC_{T-1}} - E_{T-1}[\delta^T e^{-bW_T}]\\
&= \max_{C_{T-1},\omega_{T-1}} -\delta^{T-1}e^{-bC_{T-1}} - E_{T-1}\left[\delta^T e^{-b(W_{T-1}-C_{T-1})[R_f+\omega_{T-1}(R_{r,T-1}-R_f)]}\right]\\
&= \max_{C_{T-1},\omega_{T-1}} -\delta^{T-1}e^{-bC_{T-1}} - \delta^T e^{-b(W_{T-1}-C_{T-1})\left[R_f+\omega_{T-1}\left(\overline{R}-R_f\right)\right]+\frac{1}{2}b^2(W_{T-1}-C_{T-1})^2\omega^2_{T-1}\sigma^2}
\end{aligned}$$

Maximizing this with respect to ω_{T-1} is the same as maximizing

$$\max_{\omega_{T-1}} b(W_{T-1} - C_{T-1})[R_f + \omega_{T-1}(\overline{R} - R_f)] - \frac{1}{2}b^2(W_{T-1} - C_{T-1})^2\omega^2_{T-1}\sigma^2$$

Taking the derivative with respect to ω_{T-1} we have

$$b(W_{T-1} - C_{T-1})(\overline{R} - R_f) - b^2(W_{T-1} - C_{T-1})^2\omega_{T-1}\sigma^2 = 0$$

or

$$\omega^*_{T-1} = \frac{(\overline{R} - R_f)}{(W_{T-1} - C_{T-1})b\sigma^2}$$

b. Solve for the optimal level of consumption at date $T-1$, C^*_{T-1}. C^*_{T-1} will be a function of W_{T-1}, b, ρ, R_f, $\overline{R}$, and σ^2.

Answer:

$$\max_{C_{T-1},\omega_{T-1}} -\delta^{T-1}e^{-bC_{T-1}} - \delta^T e^{-b(W_{T-1}-C_{T-1})\left[R_f+\omega_{T-1}(\overline{R}-R_f)\right]+\frac{1}{2}b^2(W_{T-1}-C_{T-1})^2\omega^2_{T-1}\sigma^2}$$

Taking the derivative of this expression with respect to C_{T-1} gives

$$\begin{aligned}
0 = {}&\delta^{T-1}be^{-bC_{T-1}} - \delta^T\{b[R_f + \omega_{T-1}(\overline{R} - R_f)] - b^2(W_{T-1} - C_{T-1})\omega^2_{T-1}\sigma^2\}\\
&\times e^{-b(W_{T-1}-C_{T-1})\left[R_f+\omega_{T-1}(\overline{R}-R_f)\right]+\frac{1}{2}b^2(W_{T-1}-C_{T-1})^2\omega^2_{T-1}\sigma^2}
\end{aligned}$$

substituting in for ω^*_{T-1} and simplifying leads to

$$\begin{aligned}
0 = {}&e^{-bC_{T-1}} - \delta\left\{\left[R_f + \frac{(\overline{R} - R_f)^2}{(W_{T-1} - C_{T-1})b\sigma^2}\right] - \frac{(\overline{R} - R_f)^2}{(W_{T-1} - C_{T-1})b\sigma^2}\right\}\\
&\times e^{-b(W_{T-1}-C_{T-1})\left[R_f+\frac{\left(\overline{R}-R_f\right)^2}{(W_{T-1}-C_{T-1})b\sigma^2}\right]+\frac{1}{2}\frac{\left(\overline{R}-R_f\right)^2}{\sigma^2}}
\end{aligned}$$

$$0 = e^{-bC_{T-1}} - \delta R_f e^{-b(W_{T-1}-C_{T-1})R_f-\frac{1}{2}\frac{\left(\overline{R}-R_f\right)^2}{\sigma^2}}$$

This implies

$$e^{-bC_{T-1}} = e^{\ln\delta + \ln R_f - b(W_{T-1} - C_{T-1})R_f - \frac{1}{2}\frac{(\bar{R}-R_f)^2}{\sigma^2}}$$

or

$$-bC_{T-1} = -\rho + \ln R_f - b\left(W_{T-1} - C_{T-1}\right)R_f - \frac{1}{2}\frac{(\bar{R}-R_f)^2}{\sigma^2}$$

Solving for C_{T-1}, we obtain

$$C^*_{T-1} = \frac{R_f}{1+R_f}W_{T-1} + H$$

where $H \equiv \left[\rho - \ln R_f + \frac{1}{2}(\bar{R}-R_f)^2/\sigma^2\right]/[b(1+R_f)]$.

c. **Solve for the indirect utility function of wealth at date $T-1$, $J(W_{T-1}, T-1)$.**

Answer:

$$\begin{aligned}
J\left(W_{T-1}, T-1\right) &= -\delta^{T-1}e^{-bC^*_{T-1}} - \delta^{T}e^{-b\left(W_{T-1}-C^*_{T-1}\right)\left[R_f+\omega^*_{T-1}\left(\bar{R}-R_f\right)\right]+\frac{1}{2}b^2\left(W_{T-1}-C^*_{T-1}\right)^2\omega^{*2}_{T-1}\sigma^2} \\
&= -\delta^{T-1}e^{-b\frac{R_f}{1+R_f}W_{T-1}-bH} - \delta^{T}e^{-b\left[\frac{1}{1+R_f}W_{T-1}-H\right]R_f-\frac{\left(\bar{R}-R_f\right)^2}{\sigma^2}+\frac{1}{2}\frac{\left(\bar{R}-R_f\right)^2}{\sigma^2}} \\
&= -\delta^{T-1}e^{-b\frac{R_f}{1+R_f}W_{T-1}-bH} - \delta^{T-1}e^{-\rho-b\frac{R_f}{1+R_f}W_{T-1}+bR_fH-\frac{1}{2}\frac{\left(\bar{R}-R_f\right)^2}{\sigma^2}} \\
&= -\delta^{T-1}e^{-b\frac{R_f}{1+R_f}W_{T-1}}\left(e^{-bH}+e^{-\rho+bR_fH-\frac{1}{2}\frac{\left(\bar{R}-R_f\right)^2}{\sigma^2}}\right) \\
&= -\delta^{T-1}G\,e^{-b\frac{R_f}{1+R_f}W_{T-1}}
\end{aligned}$$

where $G \equiv \left(e^{-bH} + e^{-\rho+bR_fH-\frac{1}{2}\frac{\left(\bar{R}-R_f\right)^2}{\sigma^2}}\right)$ **is independent of W_{T-1}.**

d. **Derive the optimal portfolio weight at date $T-2$, ω^*_{T-2}.**

Answer:

$$\begin{aligned}
&\max_{C_{T-2},\omega_{T-2}} -\delta^{T-2}e^{-bC_{T-2}} - E_{T-1}\left[\delta^{T-1}G\,e^{-b\frac{R_f}{1+R_f}W_{T-1}}\right] \\
&= \max_{C_{T-2},\omega_{T-2}} -\delta^{T-2}e^{-bC_{T-2}} - E_{T-1}\left[\delta^{T-1}G\,e^{-b\frac{R_f}{1+R_f}\left(W_{T-2}-C_{T-2}\right)\left[R_f+\omega_{T-2}\left(R_{r,T-2}-R_f\right)\right]}\right] \\
&= \max_{C_{T-2},\omega_{T-2}} -\delta^{T-2}e^{-bC_{T-2}} \\
&\quad -\delta^{T-1}G\,e^{-\frac{bR_f}{1+R_f}\left(W_{T-2}-C_{T-2}\right)\left[R_f+\omega_{T-2}\left(\bar{R}-R_f\right)\right]+\frac{1}{2}\left(\frac{bR_f}{1+R_f}\right)^2\left(W_{T-2}-C_{T-2}\right)^2\omega^2_{T-2}\sigma^2}
\end{aligned}$$

Maximizing this with respect to ω_{T-1} is the same as maximizing

$$\max_{\omega_{T-1}} \frac{bR_f}{1+R_f}\left(W_{T-2}-C_{T-2}\right)\left[R_f+\omega_{T-2}\left(\overline{R}-R_f\right)\right]-\frac{1}{2}\left(\frac{bR_f}{1+R_f}\right)^2\left(W_{T-2}-C_{T-2}\right)^2\omega_{T-2}^2\sigma^2$$

Taking the derivative with respect to ω_{T-2} we have

$$\frac{bR_f}{1+R_f}(W_{T-2}-C_{T-2})(\overline{R}-R_f)-\left(\frac{bR_f}{1+R_f}\right)^2(W_{T-2}-C_{T-2})^2\omega_{T-2}\sigma^2=0$$

or

$$\omega_{T-2}^*=\frac{(\overline{R}-R_f)(1+R_f)}{(W_{T-2}-C_{T-2})bR_f\sigma^2}$$

e. Solve for the optimal level of consumption at date $T-2$, C_{T-2}^*.

Answer:

$$\max_{C_{T-2},\omega_{T-2}} -\delta^{T-2}e^{-bC_{T-2}}-\delta^{T-1}G\,e^{-\frac{bR_f}{1+R_f}\left(W_{T-2}-C_{T-2}\right)\left[R_f+\omega_{T-2}\left(\overline{R}-R_f\right)\right]+\frac{1}{2}\left(\frac{bR_f}{1+R_f}\right)^2\left(W_{T-2}-C_{T-2}\right)^2\omega_{T-2}^2\sigma^2}$$

Taking the derivative of this expression with respect to C_{T-2} and simplifying gives

$$e^{-bC_{T-2}}-\delta G\frac{R_f^2}{1+R_f}e^{-\frac{bR_f}{1+R_f}(W_{T-2}-C_{T-2})R_f-\frac{1}{2}\frac{(\overline{R}-R_f)^2}{\sigma^2}}=0$$

This implies

$$e^{-bC_{T-2}}=e^{\ln\frac{\delta GR_f^2}{1+R_f}-\frac{bR_f^2}{1+R_f}\left(W_{T-2}-C_{T-2}\right)-\frac{1}{2}\frac{\left(\overline{R}-R_f\right)^2}{\sigma^2}}$$

or

$$-bC_{T-2}=\ln\frac{\delta GR_f^2}{1+R_f}-\frac{bR_f^2}{1+R_f}(W_{T-2}-C_{T-2})-\frac{1}{2}\frac{(\overline{R}-R_f)^2}{\sigma^2}$$

Solving for C_{T-2}, we obtain

$$C_{T-2}^*=\frac{R_f^2}{1+R_f+R_f^2}W_{T-2}+K$$

where $K\equiv\left[-\ln\frac{\delta GR_f^2}{1+R_f}+\frac{1}{2}\left(\overline{R}-R_f\right)^2/\sigma^2\right]\left(1+R_f\right)/\left[b\left(1+R_f+R_f^2\right)\right]$.

4. **An individual faces the following consumption and portfolio choice problem:**

$$\max_{C_t,\omega_t\forall t} E_0\left[\sum_{t=0}^{T-1}\delta^t \ln[C_t] + \delta^T \ln[W_T]\right]$$

where each period the individual can choose between a risk-free asses paying a time-varing return of R_{ft} over the period from t to $t+1$ and a single risky asset. The individual receives no wage income. The risky asset's return over the period from t to $t+1$ is given by

$$R_{rt} = \begin{cases} (1+u_t)R_{ft} \text{ with probability } \frac{1}{2} \\ (1+d_t)R_{ft} \text{ with probability } \frac{1}{2} \end{cases}$$

where $u_t > 0$ and $-1 < d_t < 0$. Let ω_t be the individual's proportion of wealth invested in the risky asset at date t. Solve for the individual's optimal portfolio weight ω_t^* for $t = 0, \ldots, T-1$.

Answer: **With log utility, the individual's portfolio choice is myopic. It is the same for a multi-period problem as it would be for a one-period problem. As shown in Chapter 5, the first order conditions for date t are**

$$1 = R_{ft}E_t\left[\frac{1}{R_{ft} + \omega_t(R_{rt} - R_{ft})}\right]$$

or

$$\begin{aligned} \frac{2}{R_{ft}} &= \frac{1}{R_{ft} + \omega_t u_t R_{ft}} + \frac{1}{R_{ft} + \omega_t d_t R_{ft}} \\ &= \frac{1}{R_{ft}[1+\omega_t u_t]} + \frac{1}{R_{ft}[1+\omega_t d_t]} \end{aligned}$$

Therefore

$$\begin{aligned} 2 &= \frac{2 + \omega_t(u_t + d_t)}{[1+\omega_t u_t][1+\omega_t d_t]} \\ 2(1 + \omega_t(u_t + d_t) + \omega_t^2 u_t d_t) &= 2 + \omega_t(u_t + d_t) \\ 2\omega_t(u_t + d_t) + 2\omega_t^2 u_t d_t &= \omega_t(u_t + d_t) \\ 2\omega_t^2 u_t d_t &= -\omega_t(u_t + d_t) \\ \omega_t &= -\frac{u_t + d_t}{2u_t d_t} \end{aligned}$$

Answers to Chapter 6 Exercises

1. **Two individuals agree at date 0 to a forward contract that matures at date 2. The contract is written on an underlying asset that pays a dividend at date 1 equal to D_1. Let $\tilde{f}_2$ be the date 2 random payoff (profit) to the individual who is the long party in the forward contract. Also let m_{0i} be the stochastic discount factor over the period from dates 0 to i where $i=1, 2$, and let $E_0[\cdot]$ be the expectations operator at date 0. What is the value of $E_0[m_{02}\tilde{f}_2]$? Explain your answer.**

Answer: **Let S_i be the price of the underlying asset at date i and let D_0 be the date 0 present value of dividends that it pays between dates 0 and 1. Pricing using a stochastic discount factor implies $S_0 = E_0[m_{01}D_1] + E_0[m_{02}S_2] = D_0 + E_0[m_{02}S_2]$ where D_0 is the date 0 present value of dividends. If we let F_{02} be the forward price, then we know that the payoff to the long party is $\tilde{f}_2 = S_2 - F_{02}$. This long forward position represents ownership in a share of the underlying asset, a short position (selling) the underlying asset's dividends, and borrowing an amount such that the repayment at date 2 equals F_{02}. Using the stochastic discount factor approach to pricing, we know that $E_0[m_{02}\tilde{f}_2] = E_0[m_{02}(S_2 - F_{02})] = E[m_{02}S_2] - E[m_{02}F_{02}]$. Now note that $S_0 = E_0[m_{01}D_1] + E_0[m_{02}S_2] = D_0 + E_0[m_{02}S_2]$ where D_0 is the date 0 present value of dividends. Also note that $E[m_{02}F_{02}] = E[m_{02}]F_{02} = R_f^{-2}F_{02}$. Thus we have**

$$\begin{aligned} E_0[m_{02}\tilde{f}_2] &= E[m_{02}S_2] - E[m_{02}F_{02}] \\ &= S_0 - D_0 - R_f^{-2}F_{02} \end{aligned}$$

However, we know that the absence of arbitrage implies that the forward price satisfies $F_{02} = R_f^2(S_0 - D_0)$, which implies that $E_0[m_{02}\tilde{f}_2] = 0$.

2. **Assume that there is an economy populated by infinitely lived representative individuals who maximize the lifetime utility function**

$$E_0\left[\sum_{t=0}^{\infty} -\delta^t e^{-ac_t}\right]$$

where c_t is consumption at date t and $a > 0$, $0 < \delta < 1$. The economy is a Lucas endowment economy (Lucas 1978) having multiple risky assets paying date t dividends that total d_t per capita. Write down an expression for the equilibrium per capita price of the market portfolio in terms of the assets' future dividends.

Answer: **A result of the Lucas (1978) model is that the price of a risky asset, P_0, satisfies**

$$P_0 = E_0\left[\sum_{t=1}^{\infty} \frac{U_c(c_t,t)}{U_c(c_0,0)} d_t\right]$$

In this problem $U(c_t,t) = -\delta^t e^{-ac_t}$, so that $U_c(c_t,t) = a\delta^t e^{-ac_t}$. Also, because this is an endowment economy with one share per individual, we have $c_t = d_t$. Thus,

$$P_0 = E_0\left[\sum_{t=1}^{\infty} \frac{U_c(c_t,t)}{U_c(c_0,0)} d_t\right] = E_0\left[\sum_{t=1}^{\infty} \delta^t e^{-a(d_t-d_0)} d_t\right]$$

3. For the Lucas model with labor income, show that assumptions (6.25) and (6.26) lead to the pricing relationship (6.27) and (6.28).

Answer:

$$P_t = E_t\left[\sum_{t=1}^{\infty} \delta^j \left(\frac{C^*_{t+j}}{C^*_t}\right)^{\gamma-1} d_{t+j}\right]$$

or

$$\begin{aligned} P_t/d_t &= E_t\left[\sum_{j=1}^{\infty} \delta^j \left(\frac{C^*_{t+j}}{C^*_t}\right)^{\gamma-1} \left(\frac{d_{t+j}}{d_t}\right)\right] \\ &= E_t\left[\sum_{j=1}^{\infty} \delta^j e^{(\gamma-1)\ln(C_{t+j}/C_t)+\ln(d_{t+j}/d_t)}\right] \end{aligned}$$

Now

$$\begin{aligned} \ln\left(C_{t+j}/C_t\right) &= j\cdot\mu_c + \sigma_c \sum\nolimits_{i=1}^{j} \eta_{t+j} \\ \ln\left(D_{t+j}/D_t\right) &= j\cdot\mu_d + \sigma \sum\nolimits_{i=1}^{j} \varepsilon_{t+j} \end{aligned}$$

so that

$$\begin{aligned} P_t/d_t &= E_t\left[\sum_{j=1}^{\infty} \delta^j e^{(\gamma-1)\left(j\mu_c + \sigma_c \sum_{i=1}^{j} \eta_{t+j}\right) + j\mu_d + \sigma_d \sum_{i=1}^{j} \varepsilon_{t+j}}\right] \\ &= E_t\left[\sum_{j=1}^{\infty} \delta^j e^{j[(\gamma-1)\mu_c+\mu_c] + \sum_{i=1}^{j}[(\gamma-1)\sigma_c\eta_{t+j}+\sigma_d\varepsilon_{t+j}]}\right] \\ &= \sum_{j=1}^{\infty} \delta^j e^{j[(\gamma-1)\mu_c+g_d]} e^{\frac{1}{2}j[(1-\gamma)^2\sigma_c^2+\sigma_d^2-2(1-\gamma)\sigma_c\sigma_d\rho]} \\ &= \sum_{j=1}^{\infty} e^{j[\ln\delta-(1-\gamma)\mu_c+\mu_d+\frac{1}{2}((1-\gamma)^2\sigma_c^2+\sigma_d^2)-(1-\gamma)\sigma_c\sigma_c\rho]} \\ &= \frac{1}{1-\delta e^{-(1-\gamma)\mu_c+\mu_d+\frac{1}{2}\left((1-\gamma)^2\sigma_c^2+\sigma_d^2\right)-(1-\gamma)\sigma_c\sigma_d\rho}} - 1 \end{aligned}$$

So

$$P_t = d_t \frac{\delta e^{\alpha}}{1-\delta e^{\alpha}}$$

where

$$\alpha \equiv \mu_d - (1-\gamma)\mu_c + \frac{1}{2}[(1-\gamma)^2\sigma_c^2 + \sigma_d^2] - (1-\gamma)\rho\sigma_c\sigma_d$$

4. Consider a special case of the model of rational speculative bubbles discussed in this chapter. Assume that infinitely lived investors are risk-neutral and that there is an asset paying a constant, one-period risk-free return of $R_f = \delta^{-1} > 1$. There is also an infinitely lived risky asset with price p_t at date t. The risky asset is assumed to pay a dividend of d_t that is declared at date t and paid at the end of the period, date $t+1$. Consider the price $p_t = f_t + b_t$ where

$$f_t = \sum_{i=0}^{\infty} \frac{E_t[d_{t+i}]}{R_f^{\,i+1}} 1 \tag{1}$$

and

$$b_{t+1} = \begin{cases} \dfrac{R_f}{q_t} b_t + e_{t+1} & \text{with probability } q_t \\ z_{t+1} & \text{with probability } 1-q_t \end{cases} \tag{2}$$

where $E_t[e_{t+1}] = E_t[z_{t+1}] = 0$ and where q_t is a random variable as of date $t-1$ but realized at date t and is uniformly distributed between 0 and 1.

a. Show whether or not $p_t = f_t + b_t$, subject to the specifications in (1) and (2), is a valid solution for the price of the risky asset.

Answer: We need to check that (2) satisfies $E_t[b_{t+1}] = R_f b_t$. While q_t is a random variable as of date $t-1$, it is realized (known) as of date t. Thus,

$$E_t[b_{t+1}] = \frac{R_f}{q_t} b_t q_t + E_t[e_{t+1}] q_t + (1-q_t) E_t[z_{t+1}] = R_f b_t$$

so it is a valid solution.

b. **Suppose that p_t is the price of a barrel of oil. If $p_t \geq p_{\text{solar}}$, then solar energy, which is in perfectly elastic supply, becomes an economically efficient perfect substitute for oil. Can a rational speculative bubble exist for the price of oil? Explain why or why not.**

Answer: **Since $E_t\left[b_{t+1}\right] = R_f b_t$, we see that**

$$\lim_{i\to\infty} E_t[b_{t+i}] = \begin{cases} +\infty & \text{if } b_t > 0 \\ -\infty & \text{if } b_t < 0 \end{cases} \tag{*}$$

For limited liability assets, such as oil, we cannot have a bubble path with a price becoming negative, so we need to consider only bubbles with $b_t > 0$. In this case, we see from the above equation (*) that for a bubble solution to exist, the bubble component must be expected to increase infinitely. But this cannot be a rational expectation if there is an upper bound on the price of oil, as would be the case if there was a perfect substitute in perfectly elastic supply. Thus, since p_t cannot rise above p_{solar}, b_t cannot rise above $p_{\text{solar}} - p_t^*$. Thus, a bubble path where b_t must be expected to increase to infinity cannot possibly occur.

c. **Suppose p_t is the price of a bond that matures at date $T < \infty$. In this context, the d_t for $t \leq T$ denotes the bond's coupon and principal payments. Can a rational speculative bubble exist for the price of this bond? Explain why or why not.**

Answer: **For similar reasons, a rational speculative bubble cannot exist for the price of a bond. Since, at maturity, the bond's price must be $p_T = d_T$ and zero after date T, its price cannot rationally be expected to satisfy equation (*) and increase infinitely. Thus, a bubble path is invalid, and the only rational price is $p_t = p_t^*$.**

5. **Consider an endowment economy with representative agents who maximize the following objective function:**

$$\max_{C_s\{\omega_s\},\forall s,i} E_t\left[\sum_{s=t}^{T} \delta^s u(C_s)\right]$$

where $T < \infty$. Explain why a rational speculative asset price bubble could not exist in such an economy.

Answer: **With the economy, and therefore assets (or asset markets), having a finite horizon, asset prices could not have the form $p_t = f_t + b_t$ with $b_t \neq 0$ because at date T, $p_T = f_T = d_T$ which is an asset's final dividend payment. Since $b_T = 0$ with certainty, then the bubble process $E_t[b_{t+1}] = \delta^{-1} b_t$ implies $E_{T-1}[b_T] = E_{T-1}[0] = \delta^{-1} b_{T-1}$, or $b_{T-1} = 0$. A similar argument implies $b_t = 0$ for all previous dates, $t < T - 1$.**