Chapter 7 Vector Spaces

7.1 Vector Spaces and Subspaces

Definition 7.1 A ***vector space V*** is a set with the following properties:

a) If $\mathbf{x} \in \boldsymbol{V}$, then for any scalar $c \in \mathbb{R}$, $c\mathbf{x} \in \boldsymbol{V}$.

b) If $\mathbf{x}, \mathbf{y} \in \boldsymbol{V}$, then $\mathbf{x} + \mathbf{y} \in \boldsymbol{V}$.

A vector space is closed under scalar multiplication and addition. That is, any multiple of an element in the vector space and the sum of any pair of elements are still elements in the vector space.

The set $\mathbb{R}^n$ satisfies both properties and therefore is a vector space, but $\mathbb{R}_+^n$ is not. The definition of the vector space does not specifically restrict itself to sets of vectors. The set of matrices $\mathbb{R}^{m\times n}$ is also a vector space. A vector space can be a set of any object, provided that it satisfies both of the two properties, and we do not even need to define the addition and scalar multiplication in the conventional sense. This is why the definition of vector space in other textbooks, for example **Simon & Blume** [1994], page 751, requires ten conditions. See also **Leon** [1994], page 114 for examples and page 115 for Theorem 3.1.1.

Problem Leon [1994], page 116, # 4, 6, page 127, #3, 6, 7.

4. Show that $\mathbb{R}^{m\times n}$ with the usual addition and scalar multiplication of matrices is a vector space.
6. Let $\boldsymbol{P}$ be the set of all polynomials. Show that $\boldsymbol{P}$ with the usual addition and scalar multiplication of functions forms a vector space.

Definition 7.2 A set $\boldsymbol{S}$ is a ***subspace*** of a vector space $\boldsymbol{V}$ if $\boldsymbol{S} \subseteq \boldsymbol{V}$ and $\boldsymbol{S}$ is also a vector space.

Example Let $\boldsymbol{S}$ be the set of all the solutions of a homogeneous system of linear equations $\mathbf{Ax} = \mathbf{0}$, for a given matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$. The set $\boldsymbol{S}$ is a vector space. Why? Then we can say that $\boldsymbol{S}$ is a subspace of $\mathbb{R}^n$.

Problem Show that the set $\{\mathbf{x}|\mathbf{Ax} = \mathbf{b}, \mathbf{b} \neq \mathbf{0}\}$ is not a subspace of $\mathbb{R}^n$, for a given matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$

Problem Simon & Blume [1994], page 755, #27.1, 27.2, 27.3, 27.4.

1. Which of the following are subspaces of $\mathbb{R}^2$? Explain your answer.
   a. $\{(x,y)|x=0\}$,
   b. $\{(x,y)|x=1\}$,
   c. $\{(x,y)|3x-4y=0\}$,
   d. $\{(x,y)|x^2=y^2\}$,
   e. $\{(0,1)\}$,
   f. $\{(x,y)|x+y=0, x-y=0\}$.
2. Show that the set $\mathbb{R}^2_+$ is closed under addition but is not a subspace.
3. Show that the set $\boldsymbol{W}=\{(x_1,x_2)|(x_1,x_2)\in \mathbb{R}^2, x_1=0 \text{ or } x_2=0\}$ is closed under multiplication but it is not a subspace.
4. Show that the set $\boldsymbol{V}=\{(a,b,b,c)|a,b,c\in\mathbb{R}\}$ is a subspace in $\mathbb{R}^4$.

Problem Leon [1994], page 127, #3, 8.

3. Determine whether or not the following are subspaces of $\mathbb{R}^{n\times n}$.
   a. The set of all $n\times n$ diagonal matrices.
   b. The set of all $n\times n$ lower triangular matrices.
   c. The set of all $n\times n$ matrices **A** such that $a_{12}=1$.
   d. The set of all $n\times n$ matrices **B** such that $b_{11}=0$.
   e. The set of all symmetric $n\times n$ matrices.
   f. The set of all singular $n\times n$ matrices.
8. Let **A** be a particular matrix in $\mathbb{R}^{n\times n}$. Determine whether the following are subspaces of $\mathbb{R}^{n\times n}$.
   a. $\boldsymbol{S}_1=\{\mathbf{B}|\mathbf{B}\in\mathbb{R}^{n\times n}, \mathbf{AB}=\mathbf{BA}\}$,
   b. $\boldsymbol{S}_2=\{\mathbf{B}|\mathbf{B}\in\mathbb{R}^{n\times n}, \mathbf{AB}\neq\mathbf{BA}\}$,
   c. $\boldsymbol{S}_3=\{\mathbf{B}|\mathbf{B}\in\mathbb{R}^{n\times n}, \mathbf{BA}=\mathbf{0}\}$.

7.2 Spanning Sets and Basis of Vector Space in $\mathbb{R}^n$

Definition 7.3 A vector space $\boldsymbol{V}\in\mathbb{R}^n$ is ***spanned*** or ***generated*** by vectors $\mathbf{a}_1,\mathbf{a}_2,\dots,\mathbf{a}_k\in\boldsymbol{V}$ if for any vector $\mathbf{b}\in\boldsymbol{V}$, $\mathbf{b}$ can be written as a linear combination of the vectors $\mathbf{a}_1,\mathbf{a}_2,\dots,\mathbf{a}_k$. That is, for any $\mathbf{b}\in\boldsymbol{V}$, there exists some coefficients $c_1,c_2,\dots,c_k\in\boldsymbol{V}$ such that $c_1\mathbf{a}_1+c_2\mathbf{a}_2+\cdots+c_k\mathbf{a}_k=\mathbf{b}$. The vectors

To illuminate the concept of a set of vectors spanning a vector space, we will first consider vector space in $\mathbb{R}^2$. Let $\mathbf{a}_1,\mathbf{a}_2$ be vectors in $\mathbb{R}^2$. If $\mathbf{a}_1,\mathbf{a}_2$ are linearly independent, then $\mathbf{a}_1,\mathbf{a}_2$ span the vector space $\mathbb{R}^2$, because any point $\mathbf{b}$ in $\mathbb{R}^2$ can be written as a linear combination of $\mathbf{a}_1,\mathbf{a}_2$. Why?

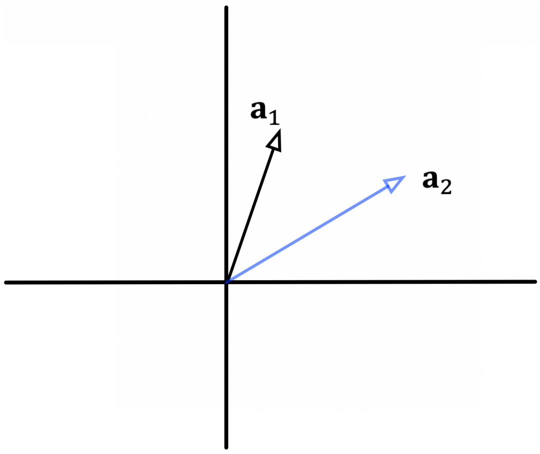

Figure 7.1 The vector $\mathbf{a}_1, \mathbf{a}_2$ span the vector space $\mathbb{R}^2$.

Problem Simon & Blume [1994], page 246, #11.9, 11.10.

9. a) Write $\begin{bmatrix}2\\2\end{bmatrix}$ as a linear combination of $\begin{bmatrix}1\\2\end{bmatrix}$ and $\begin{bmatrix}1\\4\end{bmatrix}$.

b) Write $\begin{bmatrix}1\\2\\3\end{bmatrix}$ as a linear combination of $\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}$ and $\begin{bmatrix}0\\1\\1\end{bmatrix}$.

10. Do $\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}4\\5\\12\end{bmatrix}$, and $\begin{bmatrix}0\\8\\0\end{bmatrix}$ span $\mathbb{R}^3$?

Problem Leon [1994], page 141, #15, 16.

15. Let $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_k$ be vectors that span $\boldsymbol{V}$ and let $\mathbf{a}$ be any other vector in $\boldsymbol{V}$. Show that $\mathbf{a}, \mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_k$ are linearly dependent.
16. Let $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_k$ be linearly independent vectors in a vector space $\boldsymbol{V}$. Show that $\mathbf{a}_2, \mathbf{a}_3, \dots, \mathbf{a}_k$ cannot span $\boldsymbol{V}$.

Problem Show that any two vector spaces are equal if, and only if, they have the same spanning set.

Definition 7.4 Vectors $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_k$ are a ***basis*** of the vector space $\boldsymbol{V} \subseteq \mathbb{R}^n$, if the vectors $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_k$ span the vector space $\boldsymbol{V}$ and are linearly independent.

We can write a summarizing corollary as follows.

Corollary 7.1 Let $\mathbf{A}$ be a square matrix in $\mathbb{R}^{n \times n}$. The following statements are equivalent.

1. The columns of $\mathbf{A}$ are a basis of $\mathbb{R}^n$.
2. The rows of $\mathbf{A}$ are a basis of $\mathbb{R}^n$.
3. The columns of $\mathbf{A}$ are linearly independent.
4. The rows of $\mathbf{A}$ are linearly independent.
5. $\mathbf{A}^{-1}$ exists.
6. $|\mathbf{A}| \neq 0$, i.e., $\mathbf{A}$ is nonsingular.
7. $rank\ \mathbf{A} = rank\ \mathbf{A}^\mathrm{T} = n$.
8. $\mathbf{Ax} = \mathbf{b}$ has a unique solution for any given vector $\mathbf{b} \in \mathbb{R}^n$.

Proof Exercise.❑

Example The vectors $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ in Figure 7.2 below span the vector space $\mathbb{R}^3$ but they are not a basis (Why?). However, any pair of vectors out of $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ is a basis of $\mathbb{R}^2$.

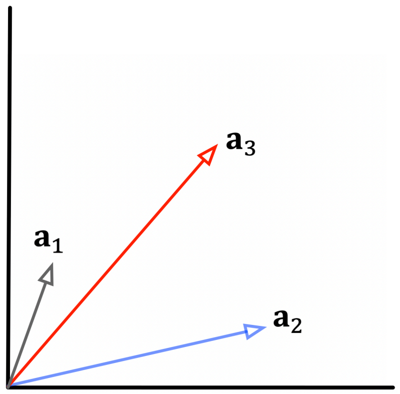

Figure 7.2 The vectors $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ span $\mathbb{R}^2$, but they are not a basis of $\mathbb{R}^2$.

Problem Simon & Blume [1994], page 249, #11.12, 11.14.

12. Which of the following are bases of $\mathbb{R}^2$?
   a) $\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}-2\\-2\end{bmatrix}$.
   b) $\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}2\\-2\end{bmatrix}$.
   c) $\begin{bmatrix}1\\-1\end{bmatrix}, \begin{bmatrix}2\\-2\end{bmatrix}$.
   d) $\begin{bmatrix}1\\-1\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}$.
14. Which of the following are bases in $\mathbb{R}^3$?
   a) $\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\2\\1\end{bmatrix}$.
   b) $\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}$.

c) $\begin{bmatrix}6\\3\\9\end{bmatrix}, \begin{bmatrix}5\\2\\8\end{bmatrix}, \begin{bmatrix}4\\1\\7\end{bmatrix}$.

d) $\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}$.

Problem Let $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_k$ be a basis of a vector space $\boldsymbol{V}$. Show that any vector $\mathbf{a} \in \boldsymbol{V}$ can be written as a unique linear combination of vectors $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_k$.

It is obvious that for a given vector space, there can be more than one set of vectors that is the basis. For example, as demonstrated in Figure 7.2 any two linearly independent vectors in $\mathbb{R}^2$ is a basis of $\mathbb{R}^2$. We can show in general that for a given vector space $\boldsymbol{V}$ that has $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_k$ as a basis, any other k linearly independent vectors in $\boldsymbol{V}$ are also a basis of $\boldsymbol{V}$.

Theorem 7.1 Let $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_k$ be a basis of a vector space $\boldsymbol{V} \subseteq \mathbb{R}^n$. Suppose that $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ are any k linearly independent vectors in $\boldsymbol{V}$. Then the vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ are also a basis of $\boldsymbol{V}$.

Proof The proof is divided into three parts:

a. Show that there exists a vector $\mathbf{x}_i \in \mathbb{R}^k$ such that $\mathbf{v}_i = \mathbf{A}\mathbf{x}_i$, for each $i = 1,2,\dots,k$, where $\mathbf{A} = [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \cdots \quad \mathbf{a}_k]$.
b. Show that $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ are linearly independent and thus the square matrix $\mathbf{X} = [\mathbf{x}_1 \quad \mathbf{x}_2 \quad \cdots \quad \mathbf{x}_k] \in \mathbb{R}^{k\times k}$ are nonsingular.
c. Show that $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ span $\boldsymbol{V}$.

(a) Since $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_k$ are a basis of $\boldsymbol{V}$, for any $\mathbf{v}_i \in \boldsymbol{V}$, it can be written as a linear combination of $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_k$. That is, for any $\mathbf{v}_i \in \boldsymbol{V}$, there exist scalars $x_{i1}, x_{i2}, \dots, x_{in}$ such that

$$
\begin{aligned}
\mathbf{v}_i &= x_{i1}\mathbf{a}_1 + x_{i2}\mathbf{a}_2 + \cdots + x_{in}\mathbf{a}_k \\
&= [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \cdots \quad \mathbf{a}_k]\begin{bmatrix}x_{i1}\\x_{i2}\\\vdots\\x_{ik}\end{bmatrix} = \mathbf{A}\mathbf{x}_i.
\end{aligned}
$$

with $\mathbf{A} = [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \cdots \quad \mathbf{a}_k]$ and $\mathbf{x}_i = \begin{bmatrix}x_{i1}\\x_{i2}\\\vdots\\x_{ik}\end{bmatrix} \in \mathbb{R}^k, i =$

$1,2,\dots,k$.

(b) Let there be a linear combination $c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_k\mathbf{x}_k = \mathbf{0}$. We have to show that such a linear combination must be trivial, i.e., $c_1 = c_2 = \cdots = c_k = 0$. From **(a)**, we have

$$\begin{aligned}\mathbf{0} &= \mathbf{A}(c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_k\mathbf{x}_k) \\ &= c_1\mathbf{A}\mathbf{x}_1 + c_2\mathbf{A}\mathbf{x}_2 + \cdots + c_k\mathbf{A}\mathbf{x}_k \\ &= c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k.\end{aligned}$$

Since $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ are linearly independent, we can conclude that $c_1 = c_2 = \cdots = c_k = 0$.

(c) From part **(a)**, we can write

$$\begin{aligned}[\mathbf{v}_1 \quad \mathbf{v}_2 \quad \cdots \quad \mathbf{v}_k] &= \mathbf{A}[\mathbf{x}_1 \quad \mathbf{x}_2 \quad \cdots \quad \mathbf{x}_k] \\ \mathbf{V} &= \mathbf{A}\mathbf{X} \\ \mathbf{A} &= \mathbf{V}\mathbf{X}^{-1}.\end{aligned}$$

where $\mathbf{X} = [\mathbf{x}_1 \quad \mathbf{x}_2 \quad \cdots \quad \mathbf{x}_k] \in \mathbb{R}^{k\times k}$. The last equality follows from Corollary 6.1 that if the columns of the square matrix $\mathbf{X}$ are linearly independent, the matrix $\mathbf{X}$ must be invertible. We need to show that for any $\mathbf{b} \in \boldsymbol{V}, \mathbf{b}$ can be written as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$. Since $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_k$ span $\boldsymbol{V}$, we can write

$$\begin{aligned}\mathbf{b} &= d_1\mathbf{a}_1 + d_2\mathbf{a}_2 + \cdots + d_k\mathbf{a}_k \\ &= \mathbf{A}\mathbf{d} \\ &= \mathbf{V}\mathbf{X}^{-1}\mathbf{d} = \mathbf{V}\mathbf{y}.\end{aligned}$$

That is, the vector $\mathbf{b}$ can be written as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$.❑

Problem: Why is $k \leq n$ in the above Theorem 7.1?

Problem: In the proof of Theorem 7.1, can the sets of vectors $\{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_k\}$ and $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ have a common element?

Theorem 7.1 implies that for a given vector space there could be many possible bases. However, we can show that of all the possible bases, they will have one thing in common: all bases of a vector space consist of the same number of vectors. This fact follows directly from the next theorem.

Theorem 7.2 Let $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_k$ be a basis of a vector space $\boldsymbol{V} \subseteq \mathbb{R}^n$. Any collection of more than k vectors in $\boldsymbol{V}$ are linearly dependent.

Proof By Lemma 6.1, it suffices to show that any set of $k + 1$ vectors must be linearly dependent. Let $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k, \mathbf{v}_{k+1}$ be any arbitrary $k + 1$ vectors in $\boldsymbol{V}$.

If any of $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k, \mathbf{v}_{k+1}$ is a zero vector, they are linearly dependent. Next, we have to show that any non-zero vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k, \mathbf{v}_{k+1} \in \boldsymbol{V}$ are linearly dependent. If $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ are linearly dependent, then also are $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k, \mathbf{v}_{k+1}$.

If $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k \in \boldsymbol{V}$ are linearly independent, by Theorem 7.1 $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ are a basis of $\boldsymbol{V}$ and since $\mathbf{v}_{k+1} \in \boldsymbol{V}$, $\mathbf{v}_{k+1}$ can be written as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$. Thus $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k, \mathbf{v}_{k+1}$ are linearly dependent by definition.❑

With this theorem, it follows directly that every basis of a vector space must have the same number of elements.

Corollary 7.2 Every basis of a vector space $\boldsymbol{V} \subseteq \mathbb{R}^n$ has equal number of elements.

Proof Exercise.❑

> **Problem** Suppose that $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ are linearly independent in a vectors space $\boldsymbol{V} \subseteq \mathbb{R}^n$ but cannot span $\boldsymbol{V}$. Show that there exists another vector $\mathbf{v}_{k+1}$ in $\boldsymbol{V}$ such that $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k, \mathbf{v}_{k+1}$ are linearly independent.

7.3 Dimension of Vector Space

We can then define the dimension of a vector space to be the number of vectors in any basis of the vector space.

Definition 7.5 The ***dimension*** of the vector space $\boldsymbol{V} \subseteq \mathbb{R}^n$, denoted dim $(\boldsymbol{V})$, is the number of the vectors in its basis.

Example Let $\mathbf{a}_1, \mathbf{a}_2$ be linearly independent vectors in $\mathbb{R}^3$. The vectors $\mathbf{a}_1, \mathbf{a}_2$ cannot span $\mathbb{R}^3$. But $\mathbf{a}_1, \mathbf{a}_2$ can span

$$\boldsymbol{L}(\mathbf{a}_1, \mathbf{a}_2) = \{\mathbf{x} | \mathbf{x} = c_1\mathbf{a}_1 + c_2\mathbf{a}_2, c_1, c_2 \in \mathbb{R}\}$$

Since $\mathbf{a}_1, \mathbf{a}_2$ are linearly independent, they are a basis for $\boldsymbol{L}(\mathbf{a}_1, \mathbf{a}_2)$, and the dimension of $\boldsymbol{L}(\mathbf{a}_1, \mathbf{a}_2)$ is 2.

Theorem 7.3 The dimension of $\mathbb{R}^n$ is *n*. That is, every basis of $\mathbb{R}^n$ has n members.

Proof Exercise.❑

Problem Let $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k$ be linearly independent vectors in a vector space $\boldsymbol{V}$. Show that the dimension of $\boldsymbol{V}$ is at least k, that is $\dim(\boldsymbol{V}) > k$.

Problem Let $\boldsymbol{V}$ be a subspace of $\boldsymbol{W}$. Show that

a. $\dim(\boldsymbol{V}) \leq \dim(\boldsymbol{W})$, and
b. $\dim(\boldsymbol{V}) < \dim(\boldsymbol{W})$, if $\boldsymbol{V} \neq \boldsymbol{W}$.

Solution a) $\dim(\boldsymbol{V}) \leq \dim(\boldsymbol{W})$. Suppose toward contradiction that $\dim(\boldsymbol{V}) > \dim(\boldsymbol{W})$. Specifically, there exist linearly independent vectors $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k$ that form a basis of $\boldsymbol{W}$, and $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{k+b}$ that form the basis of $\boldsymbol{V}$, for some positive integer b. Since $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{k+b}$ are a basis of V, they are linearly independent by definition. By Lemma 6.1, $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are also linearly independent and $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k \in \boldsymbol{V} \subseteq \boldsymbol{W}$. By Theorem 7.1, $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are also a basis of $\boldsymbol{W}$. Since $\mathbf{v}_{k+1} \in \boldsymbol{V} \subseteq \boldsymbol{W}$, $\mathbf{v}_{k+1}$ can be written as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$, and thus $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k, \mathbf{v}_{k+1}$ are linearly dependent, and by Lemma 6.1 $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{k+b}$ are linearly dependent and thus they cannot be a basis for $\boldsymbol{V}$. This is a contradiction.

b) $\dim(\boldsymbol{V}) < \dim(\boldsymbol{W})$, if $\boldsymbol{V} \neq \boldsymbol{W}$. Assume that $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are a basis of $\boldsymbol{V}$. Since $\boldsymbol{V} \subseteq \boldsymbol{W}$ and $\boldsymbol{V} \neq \boldsymbol{W}$, there exists $\mathbf{w} \in \boldsymbol{W}$ such that $\mathbf{w} \notin \boldsymbol{V}$. By the definition of vector space, the vector $\mathbf{w}$ cannot be written as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$, and thus $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k, \mathbf{w}$ are vectors in $\boldsymbol{W}$ that are linearly independent. Thus, to span $\boldsymbol{W}$, we need at least $k+1$ linearly independent vectors and the dimension of $\boldsymbol{W}$ is at least $k+1 > \dim \boldsymbol{V}$.❑

Problem Fraleigh & Beauregard [1995], page 203, #34. Let $\boldsymbol{W}$ and $\boldsymbol{U}$ be subspaces of a vector space $\boldsymbol{V}$, and let $\boldsymbol{W} \cap \boldsymbol{U} = \{\mathbf{0}\}$. Let $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k$ be a basis of $\boldsymbol{W}$ and $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m$ be a basis of $\boldsymbol{U}$. Prove that, if each vector $\mathbf{v} \in \boldsymbol{V}$ is expressible in the form $\mathbf{v} = \mathbf{w} + \mathbf{u}$ for some $\mathbf{w} \in \boldsymbol{W}$ and some $\mathbf{u} \in \boldsymbol{U}$, then $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k, \mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m$ are a basis of $\boldsymbol{V}$. Thus, $\dim(\boldsymbol{V}) = \dim(\boldsymbol{W}) + \dim(\boldsymbol{U})$.

7.4 Row, Column and Null Spaces

Associated with each matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$, we can define at least three subspaces as follows:

Definition 7.6 Let $\boldsymbol{A}$ be an $m \times n$ matrix. Define

$$Row(\mathbf{A}) = \boldsymbol{Row\ Space} \text{ of } \mathbf{A}$$
$$= \{\mathbf{c}|\mathbf{c} \text{ is a linear combination of rows of } \mathbf{A}\}$$
$$= \{\mathbf{c}|\mathbf{A}^{\mathrm{T}}\mathbf{y} = \mathbf{c}, \mathbf{y} \in \mathbb{R}^m\}$$

$$Col(\mathbf{A}) = \boldsymbol{Column\ Space} \text{ of } \mathbf{A}$$
$$= \{\mathbf{b}|\mathbf{b} \text{ is a linear combination of columns of } \mathbf{A}\}$$
$$= \{\mathbf{b}|\mathbf{A}\mathbf{x} = \mathbf{b}, \mathbf{x} \in \mathbb{R}^n\}$$

$$Null(\mathbf{A}) = \boldsymbol{Null\ Space} \text{ of } \mathbf{A}$$
$$= \{\mathbf{x}|\mathbf{x} \text{ is a solution of the homogeneous system } \mathbf{A}\mathbf{x} = \mathbf{0}\}$$
$$= \{\mathbf{x}|\mathbf{A}\mathbf{x} = \mathbf{0}, \mathbf{x} \in \mathbb{R}^n\}.$$

Problem Show that $Col(\mathbf{A}^{\mathrm{T}}) = Row(\mathbf{A})$.

Problem Show that $Row(\mathbf{A})$ and $Null(\boldsymbol{A})$ are subspaces of $\mathbb{R}^n$, and $Col(\boldsymbol{A})$ is a subspace of $\mathbb{R}^m$, for any matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$.

Problem Simon & Blume [1994], page 759, #27.7 For each of the following matrices compute bases for the row, column and null spaces.

a. $\begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix}$

b. $\begin{bmatrix} 2 & -1 & 3 \\ 4 & -2 & 5 \end{bmatrix}$,

c. $\begin{bmatrix} 2 & 1 \\ 4 & -2 \end{bmatrix}$,

d. $\begin{bmatrix} 4 & 1 & -5 & 1 \\ 8 & 5 & -10 & 8 \\ -4 & 2 & 7 & 5 \end{bmatrix}$.

Problem Leon [1994], page 166, #6. How many solutions will the linear system $\mathbf{A}\mathbf{x} = \mathbf{b}$ have if $\mathbf{b}$ is in the column space of $\mathbf{A}$ and the column vectors of $\mathbf{A}$ are linearly dependent?

7.5 Dimension of Row Space and Rank

The main result of this chapter is to show that $Row(\mathbf{A})$ and $Col(\mathbf{A})$ have the same dimension, which turns out to be equal to $rank\ \mathbf{A}$. We will also show how the dimension of $Null(\mathbf{A})$ relates to that of $Col(\mathbf{A})$. First we need the following results.

Theorem 7.4 The elementary row operations on **A** do not change its row space.

Proof There are three types of elementary row operations: multiplying a row by a nonzero scalar, interchanging the positions of any pair of rows and adding to a row a multiple of another row. It is obvious that the theorem is true for the first two types. For the last type, we can assume without loss of generality that the second row is multiplied by a constant and then added to the first row, and then show that the new set of rows still spans the same subspace $Row(\mathbf{A})$.

Let rows of **A** be denoted by $\mathbf{a}_{1\cdot}, \mathbf{a}_{2\cdot}, \dots, \mathbf{a}_{m\cdot}$ and after the elementary row operation they become $(\mathbf{a}_{1\cdot} + c\mathbf{a}_{2\cdot}), \mathbf{a}_{2\cdot}, \dots, \mathbf{a}_{m\cdot}, c \neq 0$. We will show that for any element $\mathbf{d} \in Row(\mathbf{A})$, $\mathbf{d}$ can be written as a linear combination of $(\mathbf{a}_{1\cdot} + c\mathbf{a}_{2\cdot}), \mathbf{a}_{2\cdot}, \dots, \mathbf{a}_{m\cdot}$. Since $\mathbf{d} \in Row(\mathbf{A})$, there are some constants $x_1, x_2, \dots, x_m$ such that

$$\mathbf{d} = x_1\mathbf{a}_{1\cdot} + x_2\mathbf{a}_{2\cdot} + \cdots x_m\mathbf{a}_{m\cdot}.$$

The question is whether we can find some constants $z_1, z_2, \dots, z_m$ such that

$$\begin{aligned}\mathbf{d} &= z_1(\mathbf{a}_{1\cdot} + c\mathbf{a}_{2\cdot}) + z_2\mathbf{a}_{2\cdot} + \cdots z_m\mathbf{a}_{m\cdot} \\ &= z_1\mathbf{a}_{1\cdot} + (cz_1 + z_2)\mathbf{a}_{2\cdot} + \cdots z_m\mathbf{a}_{m\cdot}\end{aligned}$$

We obtain such $z_1, z_2, \dots, z_m$ by letting $z_i = x_i, i = 1,3,4, \dots, m$ and $z_2 = x_2 - cx_1$.

The reverse to be shown is that if

$$\mathbf{d} = z_1(\mathbf{a}_{1\cdot} + c\mathbf{a}_{2\cdot}) + z_2\mathbf{a}_{2\cdot} + \cdots z_m\mathbf{a}_{m\cdot},$$

then there exist $x_1, x_2, \dots, x_m$ such that $\mathbf{d} = x_1\mathbf{a}_{1\cdot} + x_2\mathbf{a}_{2\cdot} + \cdots x_m\mathbf{a}_{m\cdot}$. This can be easily shown and is left as an exercise.❑

We now have the following corollary.

Corollary 7.3 For any matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$, $rank\,\mathbf{A} = \dim(Row(\mathbf{A}))$.

Proof Let **U** be the row echelon matrix resulting from a series of elementary row operations on **A**. Since the nonzero rows of the echelon matrix **U** are linearly independent (why?), the nonzero rows of **U** form a basis of $Row(\mathbf{U})$ and we have $rank\,\mathbf{U} = \dim\big(Row(\mathbf{U})\big)$.

Since we know from Corollary 6.3 that elementary row operation does not change number of linearly independent rows, and Theorem 6.2 that the rank of a matrix is equal to the number of linearly independent rows, we have

$$\begin{aligned} rank\ \mathbf{A} &= rank\ \mathbf{U} \\ &= \dim(Row(\mathbf{U})) \\ &= \dim\big(Row(\mathbf{A})\big). \end{aligned}$$

The last equality follows from Theorem 7.4 that $Row(\mathbf{A}) = Row(\mathbf{U})$.❑

7.6 Dimensions of Row and Column Spaces

We will show next that the dimension of the row space and column space of any matrix are equal. Corollary 7.3 above says that $rank\ \mathbf{A} = \dim(Row(\mathbf{A}))$ and when $\mathbf{A}$ is a square nonsingular matrix, Corollary 7.1, $rank\ \mathbf{A} = rank\ \mathbf{A}^{\mathrm{T}}$ and thus $\dim(Row(\mathbf{A})) = \dim(Row(\mathbf{A}^{\mathrm{T}})) = \dim\big(Col(\boldsymbol{A})\big)$. Here we will show that $\dim\big(Row(\mathbf{A})\big) = \dim(Col(\mathbf{A}))$ for any matrix $\mathbf{A}$ that does not have be square. The following follows **Leon**, [1994], Theorem 3.6.5, page 163.

Theorem 7.5 For any matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$, $\dim\big(Row(\mathbf{A})\big) = \dim(Col(\mathbf{A}))$

Proof Let $\mathbf{A}$ be a matrix of rank r, $r \le \min\{m, n\}$, whose row echelon form is $\mathbf{U}$ with r nonzero rows. The matrix $\mathbf{U}$ also contains r columns with r pivot elements. Delete the other $n - r$ columns of $\mathbf{U}$ and corresponding $n - r$ columns of $\mathbf{A}$, and call the resulting matrices $\mathbf{U}_L$ and $\mathbf{A}_L$ respectively.

We know that the columns of $\mathbf{U}_L$ are linearly independent (why?) and the homogeneous system of linear equations $\mathbf{U}_L\mathbf{x} = \mathbf{0}$ is equivalent to $\mathbf{A}_L\mathbf{x} = \mathbf{0}$ in the sense that any solution $\mathbf{x}$ of $\mathbf{U}_L\mathbf{x} = \mathbf{0}$ is also a solution of $\mathbf{A}_L\mathbf{x} = \mathbf{0}$. Since the columns of $\mathbf{U}_L$ are linearly independent, $\mathbf{x} = \mathbf{0}$ is the unique solution of $\mathbf{U}_L\mathbf{x} = \mathbf{0}$ and $\mathbf{A}_L\mathbf{x} = \mathbf{0}$. By definition, the columns of $\mathbf{A}_L$ are linearly independent. Therefore there are at least *r* columns of $\mathbf{A}$ that are linearly independent. In other words, $\dim\big(Col(\mathbf{A})\big) \ge \mathbf{r} = \dim\ (Row(\mathbf{A}))$. To show the reverse of the inequality, we write

$$\dim\big(Row(\mathbf{A})\big) = \dim\Big(Col(\mathbf{A}^{\mathrm{T}})\Big)$$

$$\geq \dim\left(Row(\mathbf{A}^{\mathrm{T}})\right)$$
$$= \dim\left(Col(\mathbf{A})\right).$$

This proves the theorem.❑

Corollary 7.4 For any matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$, $rank\ \mathbf{A}^{\mathrm{T}} = rank\ \mathbf{A}$.

Proof Immediate from Corollary 7.3 and Theorem 7.5.❑

Note that this Corollary 7.4 is stronger than Corollary 4.3 that the matrix **A** needs not be square and nonsingular.

Problem Leon [1994], page 16 page 166, #9. Let **A** and **B** be row equivalent matrices.

a. Show that the dimension of the column space of **A** equals the dimension of the column space of **B**.
b. Are the column spaces of the two matrices necessarily the same? Justify your answer.

A summarizing corollary for a square matrix **A** can be written as,

Corollary 7.5 Let **A** be a matrix in $\mathbb{R}^{n\times n}$. The following statements are equivalent.

1. $Row(\mathbf{A}) = Col(\mathbf{A}) = \mathbb{R}^n$
2. $\dim\left(Row(\mathbf{A})\right) = \dim\left(Col(\mathbf{A})\right) = n$
3. The columns of **A** are a basis of $\mathbb{R}^n$.
4. The rows of **A** are a basis of $\mathbb{R}^n$.
5. The columns of **A** are linearly independent.
6. The rows of **A** are linearly independent.
7. $\mathbf{A}^{-1}$ exists.
8. $|\mathbf{A}| \neq 0$, i.e., **A** is nonsingular.
9. $rank\ \mathbf{A} = rank\ \mathbf{A}^{\mathrm{T}} = n$.
10. $\mathbf{Ax} = \mathbf{b}$ has a unique solution for any given vector $\mathbf{b} \in \mathbb{R}^n$.

Proof Exercise.❑

7.7 Linearly Independent Rows and Row Space

For any matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$, we can state the following corollary by using Theorem 6.2 that states that the rank of matrix **A** is equal to the number of linearly independent rows.

Corollary 7.6 For any matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$, we have

$$\begin{aligned} rank\ \mathbf{A} &= rank\ \mathbf{A}^{\mathrm{T}} \\ &= \dim(Row(\mathbf{A})) = \dim(Col(\mathbf{A})) \\ &= \text{the number of linearly independent rows of } \mathbf{A} \\ &= \text{the number of linearly independent columns of } \mathbf{A} \end{aligned}$$

Proof Exercise❑

By definition, the rows of matrix **A** span $Row(\mathbf{A})$ and the columns of **A** span $Col(\mathbf{A})$. We can show in the next theorem that the set of all the linearly independent rows of **A** is a basis of $Row(\mathbf{A})$, and by the same reasoning the set of all linearly independent columns of **A** is a basis of $Col(\mathbf{A})$.

Theorem 7.6 Suppose that the rows of a matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$ can be separated such that the rows $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_k$ are linearly independent and that each of the rows $\mathbf{a}_{k+1}, \mathbf{a}_{k+2}, \dots, \mathbf{a}_m$can be written as a linear combination of rows $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_k$ Then the rows $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_k$ are a basis of $Row(\mathbf{A})$.

Proof Exercise.❑

7.8 Fundamental Theorem of Linear Algebra

The last major result of this chapter is so important that it is called the Fundamental Theorem of Linear Algebra. It shows the relationship between the dimension of the column (row) space and the null space. This proof follows **Simon & Blume** [1994], Theorem 27.10, page 777. First we will need the following Lemma 7.1 (Lemma 27.3, page 776 of **Simon & Blume** [1994]) and Lemma 7.2 (Lemma 27.4, page 776 of **Simon & Blume** [1994])

Lemma 7.1 Let $\boldsymbol{V}$ be a subspace of $\mathbb{R}^n$ with basis $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$. If $\mathbf{v}_{k+1} \in \mathbb{R}^n$ but not in $\boldsymbol{V}$, then $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k, \mathbf{v}_{k+1}$ are linearly independent.

Proof For any linear combination of $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k, \mathbf{v}_{k+1}$ that is equal to zero vector, all the coefficients must be zeroes. That is, if we have

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k + c_{k+1}\,\mathbf{v}_{k+1} = \mathbf{0}$$

we can show that $c_1 = c_2 = \cdots = c_k = c_{k+1} = 0$. Moving $c_{k+1}\,\mathbf{v}_{k+1}$ to right-hand-side, we have

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = -c_{k+1}\,\mathbf{v}_{k+1}.$$

We must have $c_{k+1} = 0$ because otherwise $\mathbf{v}_{k+1}$ can be written as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ and thus $\mathbf{v}_{k+1} \in \boldsymbol{V}$ which is a contradiction. We have

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0},$$

It follows that $c_1 = c_2 = \cdots = c_k = 0$ because $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ are linearly independent.❑

Lemma 7.2 Let $\boldsymbol{V}$ be a subspace of $\mathbb{R}^n$ with basis $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$. Then there exists vectors $\mathbf{v}_{k+1}, \mathbf{v}_{k+2}, \dots, \mathbf{v}_n$ in $\mathbb{R}^n$ such that $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k, \mathbf{v}_{k+1}, \dots, \mathbf{v}_n$ is a basis of $\mathbb{R}^n$.

Proof If $\boldsymbol{V} = \mathbb{R}^\mathrm{n}$, the lemma is trivially true. Suppose $\boldsymbol{V} \neq \mathbb{R}^n$. Then there exists a vector $\mathbf{v}_{k+1} \in \mathbb{R}^\mathrm{n}$ but not in $\boldsymbol{V}$. Thus $\mathbf{v}_{k+1}$ cannot be written as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ and thus $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k, \mathbf{v}_{k+1}$ are linearly independent by Lemma 7.1.

Let $\boldsymbol{V}'$ be the vector space spanned by $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k, \mathbf{v}_{k+1}$. If $\boldsymbol{V}' = \mathbb{R}^n$, then $n = k + 1$. Otherwise, we can repeat the argument until $\boldsymbol{V}' = \mathbb{R}^n$ and obtain $\mathbf{v}_{k+1}$, $\mathbf{v}_{k+2}$,…, $\mathbf{v}_n$ and $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k, \mathbf{v}_{k+1}, \dots, \mathbf{v}_n$ is a basis of $\mathbb{R}^n$.❑

Theorem 7.7 For any matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$,

$$\begin{aligned}\dim\big(Null(\mathbf{A})\big) &= n - rank\ \mathbf{A}\\ &= n - \dim(Col(\mathbf{A}))\\ &= n - \dim(Row(\mathbf{A})).\end{aligned}$$

Proof Let $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k$ form a basis of the subspace $Null(\mathbf{A})$. If $k = n$, then $Null(\mathbf{A}) = \mathbb{R}^n$, and it follows that $\mathbf{A} = \mathbf{0}$. (Exercise) Then $rank\ \mathbf{A} = 0$ and the theorem holds.

If $k < n$, we will show that the column space of $\mathbf{A}$ has dimension $n - k$. By Lemma 7.2, there exists $\mathbf{u}_{k+1}, \mathbf{u}_{k+2}, \dots, \mathbf{u}_n$ such that $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_{k+1}, \mathbf{u}_{k+2}, \dots, \mathbf{u}_n$ form a basis of $\mathbb{R}^n$. Then we will show that the vectors $\mathbf{Au}_{k+1}, \mathbf{Au}_{k+2}, \dots, \mathbf{Au}_n$ form a basis of $Col(\mathbf{A})$ and thus $Col(\mathbf{A})$ has dimension $n - k$.

First we will show that $\mathbf{Au}_{k+1}, \mathbf{Au}_{k+2}, \dots, \mathbf{Au}_n$ are linearly independent by showing that any linear combination of $\mathbf{Au}_{k+1}, \mathbf{Au}_{k+2}, \dots, \mathbf{Au}_n$ that is a zero vector has to be trivial. That is,

$$\begin{aligned}\mathbf{0} &= c_{k+1}\mathbf{Au}_{k+1} + c_{k+2}\mathbf{Au}_{k+2} + \cdots + c_n\mathbf{Au}_n \\ &= \mathbf{A}(c_{k+1}\mathbf{u}_{k+1} + c_{k+2}\mathbf{u}_{k+2} + \cdots + c_n\mathbf{u}_n)\end{aligned}$$

which means $c_{k+1}\mathbf{u}_{k+1} + c_{k+2}\mathbf{u}_{k+2} + \cdots + c_n\mathbf{u}_n$ is in $Null(\mathbf{A})$ and by definition can be written as a linear combination of the basis $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k$, so

$$\begin{aligned}c_{k+1}\mathbf{u}_{k+1} + c_{k+2}\mathbf{u}_{k+2} + \cdots + c_n\mathbf{u}_n \\ = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k \\ \mathbf{0} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k - c_{k+1}\mathbf{u}_{k+1} - c_{k+2}\mathbf{u}_{k+2} \\ - \cdots - c_n\mathbf{u}_n\end{aligned}$$

Since $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$ are linearly independent, $c_1 = c_2 = \cdots = c_k = c_{k+1} = c_{k+2} = \cdots = c_n = 0$ and thus $\mathbf{Au}_{k+1}, \mathbf{Au}_{k+2}, \dots, \mathbf{Au}_n$ are linearly independent.

Finally, we will show that $\mathbf{Au}_{k+1}, \mathbf{Au}_{k+2}, \dots, \mathbf{Au}_n$ span the subspace $Col(\mathbf{A})$. Let $\mathbf{b} \in Col(\mathbf{A})$, so there exists $\mathbf{x} \in \mathbb{R}^n$ such that $\mathbf{Ax} = \mathbf{b}$. Since $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$ span $\mathbb{R}^n$,

$$\begin{aligned}\mathbf{b} = \mathbf{Ax} &= \mathbf{A}(d_1\mathbf{u}_1 + d_2\mathbf{u}_2 + \cdots + d_n\mathbf{u}_n) \\ &= d_1\mathbf{Au}_1 + d_2\mathbf{Au}_2 + \cdots + d_k\mathbf{Au}_\mathrm{k} + d_{k+1}\mathbf{Au}_{k+1} + \cdots \\ &\quad + d_n\mathbf{Au}_n \\ &= d_{k+1}\mathbf{Au}_{k+1} + \cdots + d_n\mathbf{Au}_n.\end{aligned}$$

The last equation is due to the fact that $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k$ are in $Null(\mathbf{A})$. The subspace $Col(\mathbf{A})$ is then spanned by $n - k$ linearly independent vectors $\mathbf{Au}_{k+1}, \mathbf{Au}_{k+2}, \dots, \mathbf{Au}_n$ and so its dimension is also $n - k$ as needed.❑

Problem Johnson, Riess & Arnold [1998], page 164, #39, 40.

39. Prove that a square matrix **A** is nonsingular if and only if the dimension of $Null(\mathbf{A})$ is zero.
40. Let **A** be an $n \times n$ nonsingular matrix and **B** an $n \times n$ matrix. Prove that $Null(\mathbf{AB}) = Null(\mathbf{B})$ and conclude that $rank\ \mathbf{AB} = rank\ \mathbf{B}$.

Problem Simon & Blume [1994], page 771, #27.15, 27.16.

27.15 Show the following:
   a. $Col(\mathbf{AB})$ is a subspace of $Col(\mathbf{A})$ whenever **AB** is well-defined.
   b. If **B** is square and nonsingular, then $Col(\mathbf{AB}) = Col(\mathbf{A})$.

27.16 Show the following:

a) $Null(\mathbf{B})$ is a subspace of $Null(\mathbf{AB})$ whenever **AB** is well-defined.

b) If **A** is square and nonsingular, then $Null(\mathbf{AB}) = Null(\mathbf{B})$.

Problem Leon [1994], page 166, #7, 13-16.

7. Let **A** be an $m \times n$ matrix with $m > n$. Let $\mathbf{b} \in \mathbb{R}^n$ and suppose that $Null(\mathbf{A}) = \{\mathbf{0}\}$.
   a. What can you conclude about the column vectors of **A** ? Are they linearly independent? Do they span $\mathbb{R}^m$? Explain.
   b. How many solutions will the system $\mathbf{Ax} = \mathbf{b}$ have if **b** is not in the column space of **A**? How many solutions will there be if **b** is in the column space of **A**? Explain.
13. Suppose that **A** and **B** are $n \times n$ matrices with the property that $\mathbf{Ax} = \mathbf{Bx}$ for all $\mathbf{x} \in \mathbb{R}^n$. Show that
   a. $Null(\mathbf{A} - \mathbf{B}) = \mathbb{R}^n$.
   b. $\mathbf{A} - \mathbf{B}$ must have rank 0 and consequently $\mathbf{A} = \mathbf{B}$.
14. Let **A** and **B** be $n \times n$ matrices. Show that $\mathbf{AB} = \mathbf{0}$ if and only if the column space of **B** is a subspace of the null space of **A**.
15. Let $\mathbf{A} \in \mathbb{R}^{m \times n}$, $\mathbf{b} \in \mathbb{R}^m$ and let $\mathbf{x}_0$ be a particular solution to the system $\mathbf{Ax} = \mathbf{b}$. Prove the following.
   a. A vector **y** in $\mathbb{R}^n$ will be a solution to $\mathbf{Ax} = \mathbf{b}$ if and only if $\mathbf{y} = \mathbf{x}_0 + \mathbf{z}$, where $\mathbf{z} \in \mathbb{R}^n$.
   b. If $Null(\mathbf{A}) = \{\mathbf{0}\}$, then the solution $\mathbf{x}_0$ is unique.
16. Let **x** and **y** be nonzero vectors in $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively, and let $\mathbf{A} = \mathbf{xy}^{\mathrm{T}}$.
   a. Show that $\{\mathbf{x}\}$ is a basis for the column space of **A** and that $\{\mathbf{y}\}$ is a basis for the row space of **A**.
   b. What is the dimension of $Null(\mathbf{A})$?

7.9 Orthogonality between Row and Null Spaces

The next theorem shows that the $Row(\mathbf{A})$ and $Null(\mathbf{A})$ are orthogonal.

Theorem 7.8 Let $\mathbf{A}$ be a matrix of order $m \times n$. We know that both $Row(\mathbf{A})$ and $Null(\boldsymbol{A})$ are subspaces of $\mathbb{R}^n$. We have

a. If $\mathbf{c}_0 \in Row(\mathbf{A})$ and $\mathbf{x}_0 \in Null(\mathbf{A})$, then $\mathbf{c}_0^{\mathrm{T}}\mathbf{x}_0 = 0$.
b. The only common vector in $Row(\mathbf{A})$ and $Null(\mathbf{A})$ is the zero vector. That is, a vector $\mathbf{c}_0 \in Row(\mathbf{A})$ and $\mathbf{c}_0 \in Null(\mathbf{A})$ if, and only if, $\mathbf{c}_0 = \mathbf{0}$.

Proof a) Since $\mathbf{c}_0 \in Row(\mathbf{A})$, there exists $\mathbf{y} \in \mathbb{R}^m$ such that $\mathbf{c}_0 = \mathbf{A}^{\mathrm{T}}\mathbf{y}$. Since $\mathbf{x}_0 \in Null(\mathbf{A})$, we know that $\mathbf{A}\mathbf{x}_0 = \mathbf{0}$. Then, $\mathbf{c}_0^{\mathrm{T}}\mathbf{x}_0 = (\mathbf{A}^{\mathrm{T}}\mathbf{y})^{\mathrm{T}}\mathbf{x}_0 = \mathbf{y}^{\mathrm{T}}\mathbf{A}\mathbf{x}_0 = \mathbf{y}^{\mathrm{T}}\mathbf{0} = 0$.

b) From **(a)**, if $\mathbf{c}_0 \in Row(\mathbf{A})$ and $\mathbf{c}_0 \in Null(\mathbf{A})$, we have $\mathbf{c}_0^{\mathrm{T}}\mathbf{c}_0 = 0$. So $\mathbf{c}_0 = \mathbf{0}$ because the inner product of a vector is zero if, and only if, it is a zero vector. Now, if $\mathbf{c}_0 = \mathbf{0}$, then $\mathbf{c}_0 \in Row(\mathbf{A})$ because $\mathbf{A}^{\mathrm{T}}\mathbf{0} = \mathbf{0} = \mathbf{c}_0$, and $\mathbf{c}_0 \in Null(\mathbf{A})$ because $\mathbf{A}\mathbf{c}_0 = \mathbf{A}\mathbf{0} = \mathbf{0}$.❑