Macroeconomics

Lecture 2

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A Recursive Problem

- Hence, one can continue recursively in this way, solving for a collection of feedback rules of the form

$$c_t = h_t(x_t), \qquad t = T, T-1, T-2, \ldots, 0.$$

where both $c_t = h_t(x_t)$ and $x_{t+1} = f_t(x_t)$ solve these equations

$$\frac{\partial U_t(x_t,c_t)}{\partial c_t} + \frac{\partial g_t(x_t,c_t)}{\partial c_t}\left\{\frac{\partial U_{t+1}}{\partial x_{t+1}} + \frac{\partial g_{t+1}}{\partial x_{t+1}}\left[\frac{\partial U_{t+2}}{\partial x_{t+2}} + \frac{\partial g_{t+2}}{\partial x_{t+2}}\left\{\frac{\partial U_{t+3}}{\partial x_{t+3}} + \frac{\partial g_{t+3}}{\partial x_{t+3}}\left\{\ldots + \frac{\partial g_T}{\partial x_T}\left[W_0'(x_{T+1})\right]\right\}\ldots\right\}\right]\right\} = 0,$$

$$for \quad t = 0,1,\ldots,T.$$

A Recursive Problem

and

$x_{s+1} = g_s(x_s,c_s)$ for s = t,t+1,…,T.,

given that:

$c_{s+1} = h_{s+1}(x_{s+1})$ for s = t,t+1,…,T-1.

Bellman's Equations

- Define the value function for a one-period problem

$$W_1(x_T) = \max_{c_T} \{U_T(x_T, c_T) + W_0(x_{T+1})\} \qquad (1.6)$$

Subject to

$$x_{T+1} = g_T(x_T, c_T),$$
$$\text{with} \quad x_T \quad \text{given}$$

Bellman's Equations

- Form the Lagrangian, find 1st-order conditions,
- $$\frac{\partial U_T(x_T, c_T)}{\partial c_T} + \frac{\partial g_T(x_T, c_T)}{\partial c_T} W_0^{'}(x_{T+1}) = 0 \qquad (1.7)$$
- Which is the same as equation (1.5c).
- Equation (1.7) and $x_{T+1}=g_T(x_T,c_T)$ are jointly determined $c_T = h_T(x_T)$ and

$x_{T+1}=g(x_T,h_T(x_T))$. Then, put this outcome into (1.6), one has

- $$W_1(x_T) = U_T\left[x_T, h_T(x_T)\right] + W_0\left(g_T\left(x_T, h_T(x_T)\right)\right) \qquad (1.8)$$

Bellman's Equations

- Differentiating (1.8), w.r.t. x_T, gives

$$\frac{\partial W_1(x_T)}{\partial x_T} = W_1^{'}(x_T) = \left(\frac{\partial U_T}{\partial x_T} + \frac{\partial g_T}{\partial x_T} W_0^{'}(x_{T+1}) \right) + \frac{\partial h_T}{\partial x_T} \left[\frac{\partial U_T}{\partial c_T} + \frac{\partial g_T}{\partial c_T} W_0^{'}(x_{T+1}) \right]$$

- Then, together with (1.7), one has

$$W_1^{'}(x_T) = \frac{\partial U_T(x_T, h_T(x_T))}{\partial x_T} + \frac{\partial g_T(x_T, h_T(x_T))}{\partial x_T} \left[W_0^{'}\left(g_T\left(x_T, h_T(x_T)\right)\right) \right]$$

- (1.9)

Bellman's Equations

- Next, repeat the process with the value function for the two-period problem,
- $$W_2(x_{T-1}) = \max_{c_{T-1}} \{U_{T-1}(x_{T-1}, c_{T-1}) + W_1(x_T)\})$$
- where
$$W_1(x_T) = \max_{c_T} \{U_T(x_T, c_T) + W_0(x_{T+1})\}$$
- subject to
$$x_T = g_{T-1}(x_{T-1}, c_{T-1}),$$
$$with \quad x_{T-1} \quad given$$

Bellman's Equations

- Again, by repeating the same procedure, we
- $$W_2^{'}(x_{T-1}) = \frac{\partial U_{T-1}(x_{T-1}, h_{T-1}(x_{T-1}))}{\partial x_{T-1}} + \frac{\partial U_{T-1}(x_{T-1}, h_{T-1}(x_{T-1}))}{\partial h_{T-1}(x_{T-1})} \cdot \frac{\partial h_{T-1}(x_{T-1})}{\partial x_{T-1}}$$
- $$+ \frac{\partial g_{T-1}(x_{T-1}, h_{T-1}(x_{T-1}))}{\partial x_{T-1}} W_1^{'}(g_{T-1}(x_{T-1}, h_{T-1}(x_{T-1}))) \qquad (1.11)$$
- Applying (1.9) into (1.11), one has

$$W_2^{'}(x_{T-1}) = \frac{\partial U_{T-1}(x_{T-1}, h_{T-1}(x_{T-1}))}{\partial x_{T-1}} + \frac{\partial U_{T-1}(x_{T-1}, h_{T-1}(x_{T-1}))}{\partial h_{T-1}(x_{T-1})} \cdot \frac{\partial h_{T-1}(x_{T-1})}{\partial x_{T-1}}$$

$$+ \frac{\partial g_{T-1}(x_{T-1}, h_{T-1}(x_{T-1}))}{\partial x_{T-1}} \cdot \left\{ \frac{\partial U_T}{\partial x_T} + \frac{\partial g_T}{\partial x_T} \left[W_0^{'}(x_{T+1}) \right] \right\}, \qquad (1.12)$$

- where $x_T = g_{T-1}(x_{T-1}, h_{T-1}(x_{T-1}))$

Bellman's Equations

- Note that, for j = 1, equation (1.12) becomes

$$W'_{j+1}\left(x_{T-j}\right)=\frac{\partial U_{T-j}\left(x_{T-j},h_{T-j}\left(x_{T-j}\right)\right)}{\partial x_{T-j}}+\frac{\partial U_{T-j}\left(x_{T-j},h_{T-j}\left(x_{T-j}\right)\right)}{\partial h_{T-j}\left(x_{T-j}\right)}.\frac{\partial h_{T-j}\left(x_{T-j}\right)}{\partial x_{T-j}}$$

$$+\frac{\partial g_{T-j}\left(x_{T-j},h_{T-j}\left(x_{T-j}\right)\right)}{\partial x_{T-j}}.\left\{\frac{\partial U_{T-j+1}}{\partial x_{T-j+1}}+\frac{\partial g_{T-j+1}}{\partial x_{T-j+1}}W'_{j}\left(x_{T-j+1}\right)\right\},$$

(1.13)

where $x_{T-j+1}=g_{T-j}\left(x_{T-j},h_{T-j}\left(x_{T-j}\right)\right)$

Bellman's Equations

- By iteration, one has the Bellman's equation as
- $$W_{j+1}\left(x_{T-j}\right)=\max_{c_{T-j}}\left\{U_{T-j}\left(x_{T-j},c_{T-j}\right)+W_j\left(x_{T-j+1}\right)\right\} \qquad (1.14)$$

- subject to

$$x_{T-j+1}=g_{T-j}\left(x_{T-j},c_{T-j}\right),$$
$$\textit{with} \quad x_{T-j} \quad \textit{given}$$

Bellman's Equations

- The idea is to proceed recursively and to work backward.
- First solve the 1-period problem with $j+1=1$, deducing $W_1(x_T)$.(or eq.(1.6))
- Then solve the 2-period problem with $j+1=2$, deducing $W_2(x_{T-1})$. (or eq.(1.10))
- The process is repeated until we have the (T+1)-period value function, $W_{T+1}(x_0)$, and along the way we have the optimal feedback rules $c_{T-j}=h_{T-j}(x_{T-j}), j = 0,1,\dots,T$.

Bellman's Equations

- The (T+1)-period value function is;

$$W_{T+1}(x_0) = \max_{c_0}\left[U_0(x_0,c_0) + \max_{c_1}\left[U_1(x_1,c_1) + \ldots.\max_{c_T}\left[U_T(x_T,c_T) + W_0(x_{T+1})\right]\right]\right]$$

$$= \max_{c_0}\left[U_0(x_0,c_0) + W_T(x_1)\right]$$

$$\textit{Subject to} \quad x_1 = g_0(x_0,c_0), \qquad x_0 \ \textit{given.}$$

Bellman's Equations

- The first-order condition of the above problem w.r.t. u_0 is,

$$\frac{\partial U_0\left(x_0, c_0\right)}{\partial c_0} + \frac{\partial g_0\left(x_0, c_0\right)}{\partial c_0} W_T^{'}\left(x_1\right) = 0 \qquad (1.15)$$

- Rewrite equation(1.14) for j = T-1, and put into eq.(1.15), we have

$$\frac{\partial U_t\left(x_t, c_t\right)}{\partial c_t} + \frac{\partial g_t\left(x_t, c_t\right)}{\partial c_t}\left\{\frac{\partial U_{t+1}}{\partial x_{t+1}} + \frac{\partial g_{t+1}}{\partial x_{t+1}}\left[\frac{\partial U_{t+2}}{\partial x_{t+2}} + \frac{\partial g_{t+2}}{\partial x_{t+2}}\left\{\frac{\partial U_{t+3}}{\partial x_{t+3}} + \frac{\partial g_{t+3}}{\partial x_{t+3}}\left\{\ldots + \frac{\partial g_T}{\partial x_T}\left[W_0^{'}\left(x_{T+1}\right)\right]\right\}\ldots\right\}\right]\right\} = 0,$$
$$for \quad t = 0, 1, \ldots, T.$$

Bellman's Equations

- This backward recursion generates the same marginal conditions as the problem (1.5a)
- The derivation of the value functions obeys the recursion (see eq.(13) & eq. (1.9), for j=1)

$$W'_{j+1}\left(x_{T-j}\right)=\frac{\partial U_{T-j}\left[x_{T-j},h_{T-j}\left(x_{T-j}\right)\right]}{\partial x_{T-j}}+\frac{\partial U_{T-j}\left(x_{T-j},h_{T-j}\left(x_{T-j}\right)\right)}{\partial h_{T-j}\left(x_{T-j}\right)}\cdot\frac{\partial h_{T-j}\left(x_{T-j}\right)}{\partial x_{T-j}}$$

$$+\frac{\partial g_{T-j}}{\partial x_{T-j}}W'_{j}\left(g_{T-j}\left[x_{T-j},h_{T-j}\left(x_{T-j}\right)\right]\right)$$

- Comparing this equation with (1.4bbb) and (1.4ccc), we find that:

Bellman's Equations

- $W'_j(x_{T+1-j}) = \lambda_{T-j}$,
- and $W'_{j+1}(x_{T-j}) = \lambda_{T-j-1}$,
- Hence,

$$\lambda_{T-(j+1)} = \frac{\partial U_{T-j}\left[x_{T-j}, h_{T-j}\left(x_{T-j}\right)\right]}{\partial x_{T-j}} + \frac{\partial U_{T-j}\left(x_{T-j}, h_{T-j}\left(x_{T-j}\right)\right)}{\partial h_{T-j}\left(x_{T-j}\right)} . \frac{\partial h_{T-j}\left(x_{T-j}\right)}{\partial x_{T-j}} + \frac{\partial g_{T-j}}{\partial x_{T-j}} \lambda_{T-j}$$

Summary

- Define the (T+1)-period value function

$$W_{T+1}(x_0) = \max_{c_0,\dots,c_T} \left[U_0(x_0,c_0) + U_1(x_1,c_1) + \dots + U_T(x_T,c_T) + W_0(x_{T+1}) \right] \qquad (1.16)$$

$$\textit{Subject to} \qquad x_{t+1} = g_t(x_t,c_t), \qquad \textit{for all } t = 0,1,\dots,; \quad x_0 \textit{ given.}$$

Problem (1.16) is the same as

$$W_{T+1}(x_0) = \max_{c_0}\left[U_0(x_0,c_0) + \max_{c_1}\left[U_1(x_1,c_1) + \dots \max_{c_T}\left[U_T(x_T,c_T) + W_0(x_{T+1}) \right]\right]\right] \quad (1.17)$$

$$= \max_{c_0}\left[U_0(x_0,c_0) + W_T(x_1) \right]$$

$$\textit{subject to} \quad x_1 = g_0(x_0,c_0), \qquad x_0 \textit{ given.}$$

Summary

Starting to solve problem (1.16) from $t = T$,

(1) Solving for $c_T = h_T(x_T)$ to optimize $W_1(x_T)$;

(2) Solving for $c_{T-1} = h_{T-1}(x_{T-1})$ to optimize $W_2(x_{T-1})$;

(3) Continue to repeat this process until $t=0$.