Chapter 3
Calculus of Single Variable: Applications

3.1 Labor Union In a given industry, the labor union is effectively the monopoly of labor input. The market demand for the labor is a negatively sloped as shown in the figure below.

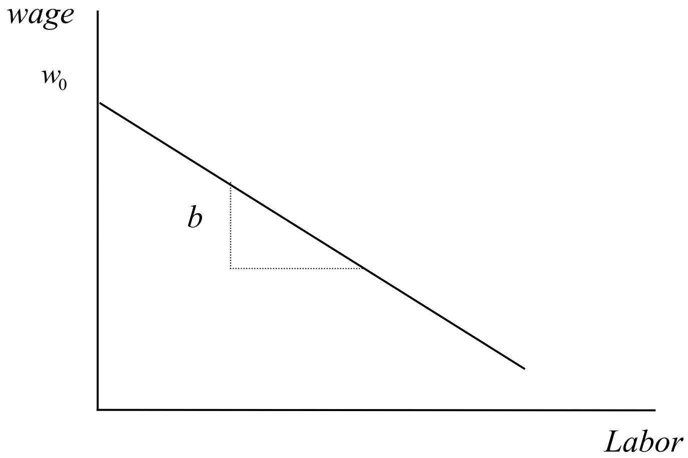

Figure 3.1 Demand for union workers

where

$w = w_0 - bL$
$=$ wages earned by union members in a given industry employing L

There are 4 possible goals the union can have:

1. max wage rate
2. max level of employment
3. max total income of employed workers
4. max rents earned by employed workers

1) Max Wage Rate: Optimal solution $w^* = w_0 = w_1$, and so $L = 0$ and it is a corner solution—a solution where the sufficient conditions are not met.

2) Max Employment: Let

$\overline{w} =$ wages earned by union members working in other industry or receiving unemployment compensation.

Then, optimal solution $w^* = \overline{w} = w_2$. That is, the union will set the wage at the level the worker can get at the other best paying industry.

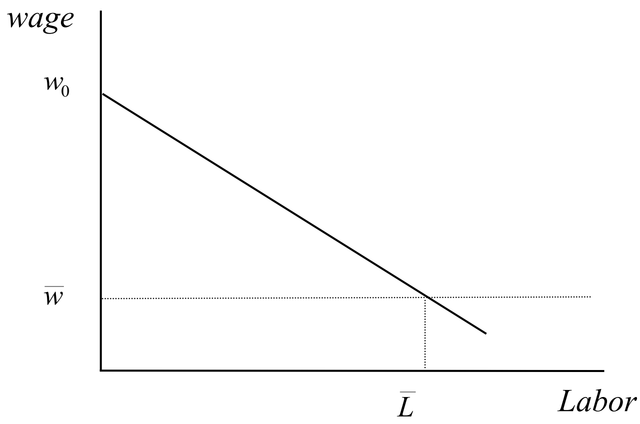

Figure 3.2 The best wage rate to set if the union maximizes employment

3) Max Total Income of Employed Union Workers:

Let I be the total income earned by all employed union workers. We can see that the total income depends on wage rate w, and so we write $I = I(w)$

$$\begin{aligned} I(w) &= wL \\ &= w\frac{w_0 - w}{b}, \; w = w_0 - bL. \end{aligned}$$

By the first-order sufficient condition,

$$I'(w^*) = \frac{w_0 - 2w^*}{b} = 0,$$

and we have the critical point $w^* = \frac{w_0}{2}$. We can test that this critical point satisfies the second-order sufficient condition

$$I''(w^*) = \frac{-2}{b} < 0.$$

So the optimal wage rate to set is $w_3 = \boldsymbol{max}\left\{\frac{w_0}{2}, \overline{w}\right\}$.

4) <u>Max Economic Rent</u>: We define rent as

$$r = w - \overline{w},$$

i.e., the difference between the wage received and the highest wage earned in other industry. The total rent earned is

$$R(w) = rL = (w - \overline{w})\frac{w_0 - w}{b},$$

first-order sufficient condition,

$$R'(w^*) = \frac{\overline{w} + w_0 - 2w^*}{b} = 0,$$

and the critical point is

$$w^* = \frac{\overline{w} + w_0}{2}.$$

The second-order sufficient condition is given by

$$R''(w^*) = -\frac{2}{b} < 0.$$

We will have the optimal solution

$$\begin{aligned} w_4 &= \frac{\overline{w} + w_0}{2} \\ &= w_3 + \frac{\overline{w}}{2},\ \text{if } w_3 > \overline{w}. \end{aligned}$$

Are w_3 and w_4 strict global maximum points? In summary, we can draw a graph to show the relation of optimal wage rates of four cases as

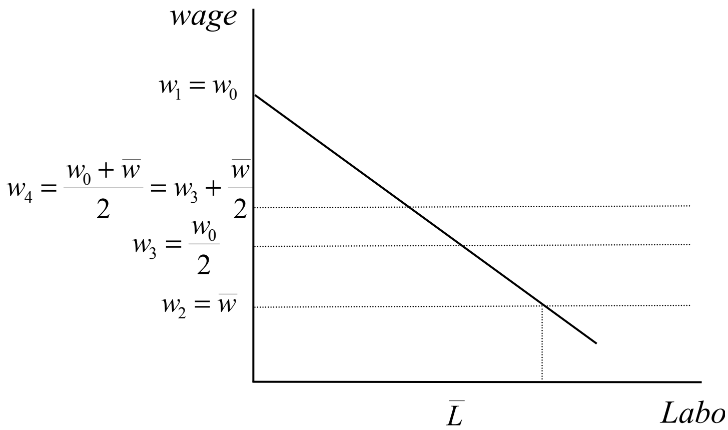

Figure 3.3 Optimal wages for 4 different objectives

HW Maximize contribution from the union workers to the union if the contribution is
a) $X per worker
b) x% of the wage earned by each worker

HW: p. 61 #2.1, 2.2, 2.3

3.2 Profit Maximization in Perfect Competition

A competitive firm with production function

$$Q = f(L, K) = 2L^{0.5}K^{0.5}.$$

Assume short-run production with $K = K_0$. The profit now has only L as decision variable.

$$\begin{aligned} \boldsymbol{max}\, \pi(L) &= pQ - wL - rK_0 \\ &= p2L^{0.5}K_0^{0.5} - wL - rK_0 \end{aligned}$$

By first-order sufficient condition,

$$\pi'(L^*) = pL^{*^{-0.5}}K_0^{0.5} - w = 0,$$

we have the critical point

$$L^* = \frac{p^2 K_0}{w^2},$$

independent of r. This can be considered the short-run demand for labor.

This critical point is the maximum point because it satisfies the second-order sufficient condition

$$\pi''\left(L^*\right) = -0.5pL^{*^{-1.5}}K_0^{0.5} < 0.$$

The optimal output is $Q = 2L^{*^{0.5}}K_0^{0.5} = \frac{2pK_0}{w}$, which is a strict global maximum point.

HW Compute the demand elasticity of the short run demand for labor $L^* = \frac{p^2 K_0}{w^2}$.

HW Redo this example with $Q = AL^{\alpha}K_0^{\beta}$. Is there any condition on the values of α and/or β?

HW Baldani, p. 62, #2.4.

3.3 Profit Maximization of a Monopoly

A monopoly faces the demand function

$$p = a - bQ,$$

and has the total cost function

$$TC = f + cQ.$$

We have,

$$\pi(Q) = TR - TC = (a - bQ)Q - f - cQ$$
$$\pi'\left(Q^*\right) = MR - MC = a - 2bQ^* - c = 0$$
$$\pi''\left(Q^*\right) = -2b < 0$$
$$Q^* = \frac{a-c}{2b}$$
$$p^* = a - bQ^* = \frac{a+c}{2}.$$

Note the optimal solution is independent of the fixed cost *f*.

HW Baldani, p.62, #2.6, 2.7

3.4 Taxation on Monopoly

Specific tax \$$t$ per unit sold is imposed on a monopoly. The tax collected is given by.

$$T = tQ .$$

The profit of the monopoly, assuming a linear demand function, is given by

$$\begin{aligned}\pi(Q) &= TR - TC - T = (a - bQ)Q - f - cQ - tQ \\ &= (a - bQ)Q - (c + t)Q - f.\end{aligned}$$

Maximizing this profit, the first- and second-order sufficient condtions are given by

$$\begin{aligned}\pi'(Q^*) &= MR - MC - t = a - 2bQ^* - (c + t) = 0 \\ \pi''(Q^*) &= -2b < 0.\end{aligned}$$

The critical point obtained from the first-order sufficient condition $Q^* = \dfrac{a - c - t}{2b}$, assuming $a - c - t > 0$, at the price $p^* = a - bQ^* = \dfrac{a + c + t}{2}$ and the maximal profit is $\pi(Q^*) = \dfrac{(a - c - t)^2}{4b} - f$.

We can perform the Comparative Static Analysis (Sensivitity Analysis). That is, how the optimal solution changes when a parameter changes,

$$\begin{aligned}\frac{dQ^*}{dt} &= \frac{-1}{2b} < 0 \\ \frac{dp^*}{dt} &= \frac{1}{2}.\end{aligned}$$

We also see how the objective function values changes when a parameter changes,

$$\frac{d\pi\left(Q^*;t\right)}{dt} = -\frac{a-c-t}{2b} < 0.$$

We will see in later chapters that the general theory for sensitivity analyses are based on the Implicit Function Theorem and Envelope Theorem.

HW What is the specific tax rate t that maximization tax revenue $T = tQ^* = t\frac{a-c-t}{2b}$?

HW Baldani, p. 62, #2.8. Redo all above when the tax is ad valorem $T = tpQ$.

3.5 Profit Maximization of Duopoly (Cournot Model)

A market with two sellers face a common demand function

$$P = a - bQ\,,$$

where $Q = q_1 + q_2$, the sum of quantity sold by both firms. The profit functions of the two firms are given respectively by

$$\begin{aligned}\pi_1\left(q_1\right) &= TR_1 - TC_1 \\ &= aq_1 - bq_1q_2 - bq_1^2 - cq_1 \\ \pi_2\left(q_2\right) &= aq_2 - bq_1q_2 - bq_2^2 - cq_2.\end{aligned}$$

Each firm is assumed to maximize its profit by assuming a given production quantity of the other's. The first-order necessary conditions product the so-called reaction functions as follows.

$$\pi_1'\left(q_1^*\right) = a - bq_2 - bq_1^* - c = 0 \Rightarrow q_1^* = \frac{a-c}{2b} - \frac{1}{2}q_2$$
$$\pi_2'\left(q_2^*\right) = a - bq_2^* - bq_1 - c = 0 \Rightarrow q_2^* = \frac{a-c}{2b} - \frac{1}{2}q_1.$$

Solving the two linear reaction functions, we have the equilibrium

$$q_1^* = \frac{a-c}{3b} = q_2^*,$$

This is called a *Nash Equilibrium* because once it is in the equilibrium neither party sees incentive to change.

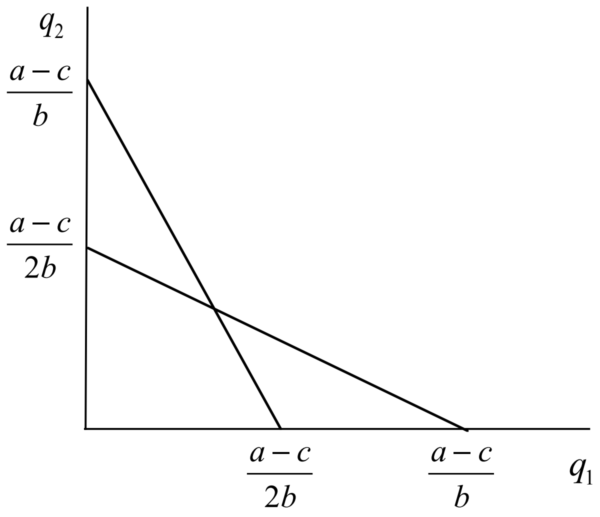

Figure 3.4 Reaction functions of two identical firms in a duopoly market.

The market quantity and price are given by

$$Q^* = q_1^* + q_2^* = 2\left(\frac{a-c}{3b}\right)$$

$$P^* = a - bQ^* = \frac{a+2c}{3}.$$

HW Redo the Duopoly model with one of the following changes

a) The two firms have different costs $c_1 \neq c_2$.

b) Each firm has a fixed cost.

HW Baldani, p. 63, #2.11.

3.6 Balanced-Budget Multiplier

Given a simple macroeconomic model as given below, when the government chooses to have a balance budget, the multipler is equal to 1.

$$Y = C + I + G$$
$$C = a + bY_d,\ 0 < b < 1$$
$$Y_d = Y - T$$
$$I = I_0 - er,\ e > 0$$

By substitution, the equilibrium income is

$$Y = a + b\left(Y - T\right) + I_0 - er + G$$
$$Y^* = \frac{a - bT + I_0 - er + G}{1 - b}.$$

Taking derivative with respect to the government expenditures and tax, we have

$$\frac{\partial Y^*}{\partial G} = \frac{1}{1-b} > 0$$
$$\frac{\partial Y^*}{\partial T} = -\frac{b}{1-b} < 0.$$

The Balanced Budget Multiplier

$$\frac{\partial Y^*}{\partial G} + \frac{\partial Y^*}{\partial T} = \frac{1}{1-b} - \frac{b}{1-b} = 1.$$

We will see a more rigorous treatment in Chapter 5.

HW Baldani, p.64, #2.15.