Assignment 3 EE320 (Section Aj. Kittichai)

Due on Nov., 15th 2020

Instruction

1) All groups must attempt question 0.
2) Odd-numbered group must attempt all odd-numbered questions.
3) To submit your homework, write your filename as follow **hw3_Group_0x**. One point will be deducted if you don't follow the format of suggested filename.

Question 0:

Consider the function f defined for all (x,y) such that

$$f\ (x, y; a) = \frac{1}{2}x^2 - x + ay(x-1) - \frac{1}{3}y^3 + a^2y^2,$$

where a is a constant.

a. Prove that $(x^*, y^*) = (1 - a^3, a^2)$ is a stationary point of $f\ (x, y; a)$.
b. State the condition under which the above stationary point is a global maximum.
c. Given that $G(a) = f\ (x^*, y^*; a)$, show the derivative of G with respect to a.
d. Calculate $\frac{\partial f\ (x,y;a)}{\partial a}$ and evaluate its value where $(x^*, y^*) = (1 - a^3, a^2)$. Compare your answer with the answer obtained in b. Are they the same?
e. Determine the domain of (x,y) in the xy-plan where f is convex.

Question 1

Consider a linear demand equation faced by a monopolist.

$$Q = 2000 + 4\sqrt{A} - 20P,$$

where A is the dollar amount of the expenditure on advertisement, P is the unit price, and Q is the amount of quantity demanded by consumers. The demand equation is an extended version of the traditional one. The proposed function captures an idea that advertising can boost the total demand in the market as potential customers get more information about the product.

Suppose that the monopolist cost function is given by $C(Q, A) = c(Q) + A$ where $c(Q) = 2Q + 1000$

Consider the following problem.

a) Construct the profit function.
b) Determine the optimal pricing (P*) and optimal advertisement (A*) that maximize the profit of the monopolist.
c) Confirm the result with the second-order differential test, i.e. hessian-matrix method.
d) What is the maximum level of profit?

Question 2: Suppose that there are three groups of people who take Sky train to commute in Bangkok. The first group is students (s), the second group is senior citizens (old), and the third group is working-aged people. The demand for each group is given by the following equations:

Demand of students: $P_s = 8 - \left(\frac{1}{2}\right) Q_s$

Demand of senior citizens: $P_{old} = 16 - 2Q_{old}$

Demand of working-aged people: $P_w = 20 - Q_w$

The Sky train operator has a constant marginal cost at MC = $4, and total cost at TC = 4Q + 10. Consider the following problems.

a) Determine the profit-maximizing level of output/price under third-degree price discrimination. Calculate the level of maximized profit.

b) Confirm your result in (a) with the second-order derivative test.

c) Calculate (i) consumer surplus for each of the three groups of consumers, and (ii) producer surplus.

d) Calculate the optimal level of total output if the Sky train operator can practice the first-degree price discrimination for each group of the consumers in the market.

Question 3:

Consider a simple two-market model where demand and supply for each market is given below. (Notationally, let's name the two markets as A and B, respectively.)

Market A:

Demand: $p_A = 10 - 2Q_A$

Supply: $p_A = 1 + Q_A$

Market B:

Demand: $p_B = 20 - Q_B$

Supply: $p_B = 2 + 2Q_B$

a. Derive the market equilibrium
b. Suppose the government imposes unit tax on consumers in both markets at the rate of $t_A$and t_B. Solve for the after-tax equilibrium as the function of t_A and t_B.
c. How much revenue can the government collect from the taxation?
d. Determine the level of t_A and t_B that maximizes government's revenue.
e. Use the second-order conditions test and show that the answer obtained in "d" is a global solution.

Question 4:

Suppose that the output *Q* of a firm depends on two inputs of the quantities *K* and *L*. The output level is determined by the production function

$$Q = 36K + 16L - 3K^2 - 2KL - L^2$$

a. Is the firm's production function strictly concave? Explain.
b. Determine the optimal input (K*, L*) that maximizes the output level.
c. Write down the firm's profit function when the price of *Q* is *P* and the per-unit factor prices of *K* and *L* are *r* and *w*, respectively, where both *r* and *w* are positive numbers. Find the levels of $K^*$and L^* that maximize the firm's profits.
d. Verify that the second-order sufficient conditions for maximum profits are satisfied.
e. Determine the effect of an increase in *r* on the firm's use of each input. (i.e. determine $\frac{\partial K^*}{\partial r}$ and $\frac{\partial L^*}{\partial r}$).

Question 0:

Consider the function f defined for all (x,y) such that

$$f\,(x, y; a) = \frac{1}{2}x^2 - x + ay(x-1) - \frac{1}{3}y^3 + a^2y^2,$$

where a is a constant.

a. Prove that $(x^*, y^*) = (1-a^3, a^2)$ is a stationary point of $f\,(x, y; a)$.
b. State the condition under which the above stationary point is a global maximum.
c. Given that $G(a) = f\,(x^*, y^*; a)$, show the derivative of G with respect to a.
d. Calculate $\frac{\partial f\,(x,y;a)}{\partial a}$ and evaluate its value where $(x^*, y^*) = (1-a^3, a^2)$. Compare your answer with the answer obtained in b. Are they the same?
e. Determine the domain of (x,y) in the xy-plan where f is convex.

a) $\frac{\partial f}{\partial x} = x - 1 + ay$ ①

$ax - a + a^2y$ ① $\times a$ = ①'

$\frac{\partial f}{\partial y} = a(x-1) - y^2 + 2a^2y$ ②

$= ax - a - y^2 + 2a^2y$

①' − ② : $y^2 - a^2y$

$y(y - a^2) = 0$ $y = a^2$ #

$x = 1 - a^3$ #

$x - 1 + a(a^2) = 0$

$x = 1 - a^3$

$H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} 1 & a \\ a & -2y+2a^2 \end{bmatrix}$

$H_1 = 1 > 0$

$H_2 = -2y + 2a^2 - a^2$

$= -2y + a^2$

$= -2a^2 + a^2$

$= -a^2 < 0$

b) Global Maximum are not possible under this function no matter what the stationary point is, since H_1 is a positive constant. Hence we will never experience negative definite with this function. #

c) $G(a) = f(x^*, y^*; a) = \frac{1}{2}(1-a^3)^2 - (1-a^3) + a(a^2)(1-a^3-1) - \frac{1}{3}(a^2)^3 + a^2(a^2)^2$

$= \frac{1}{2}(1 - 2a^3 + a^6) - 1 + a^3 + a^3 - a^6 - a^3 - \frac{1}{3}(a^6) + a^6$

$\frac{\partial G}{\partial a} = -3a^2 + 3a^5 + 3a^2 + 3a^2 - 6a^5 - 3a^2 - 2a^5 + 6a^5$

$= -3a^5 + 4a^5$

$= a^5$ #

d) $\frac{\partial f(x,y;a)}{\partial a} = y(x-1) + 2ay^2$

$= a^2(1 - a^3 - 1) + 2a(a^2)^2$

$= -a^5 + 2a^5$

$= a^5$ #

e) $|H_1| = 1 > 0$ $|H_2| = -2y + a^2 > 0$

$y < \frac{a^2}{2}$

$x = (-\infty, \infty)$

$y = (-\infty, \frac{a^2}{2})$ #

Question 1

Consider a linear demand equation faced by a monopolist.

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where A is the dollar amount of the expenditure on advertisement, P is the unit price, and Q is the amount of quantity demanded by consumers. The demand equation is an extended version of the traditional one. The proposed function captures an idea that advertising can boost the total demand in the market as potential customers get more information about the product.

Suppose that the monopolist cost function is given by $C(Q, A) = c(Q) + A$ where $c(Q) = 2Q + 1000$

Consider the following problem.

a) Construct the profit function.
b) Determine the optimal pricing (P*) and optimal advertisement (A*) that maximize the profit of the monopolist.
c) Confirm the result with the second-order differential test, i.e. hessian-matrix method.
d) What is the maximum level of profit?

a)
$$\begin{aligned}\pi &= P\cdot Q - C(Q, A)\\ &= P(2000+4\sqrt{A}-20P) - c(Q) - A\\ &= P(2000+4\sqrt{A}-20P) - 2Q - 1000 - A\\ &= P(2000+4\sqrt{A}-20P) - 2(2000+4\sqrt{A}-20P) - 1000 - A\\ &= (2000+4\sqrt{A}-20P)(P-2) - 1000 - A\end{aligned}$$

<u>Ans</u> profit function is $\pi = (2000+4\sqrt{A}-20P)(P-2) - 1000 - A$

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b) $\frac{\partial \pi}{\partial P} = 2000 + 4\sqrt{A} - 40P + 40 = 0$

$2040 + 4\sqrt{A} - 40P = 0 \longrightarrow \frac{2040+4\sqrt{A}}{40} = P \rightarrow P = 51 + \frac{\sqrt{A}}{10}$ —①

$$\frac{\partial \pi}{\partial A} = 2A^{-1/2}P - 4A^{-1/2} - 1$$

$$= \frac{2P-4}{\sqrt{A}} - 1 = 0 \quad \text{—②}$$

take ① into ② ; $\frac{2\left(51+\frac{\sqrt{A}}{10}\right)-4}{\sqrt{A}} - 1 = 0$

$$102 + \frac{\sqrt{A}}{5} - 4 = \sqrt{A}$$

$$510 + \sqrt{A} - 20 = 5\sqrt{A}$$

$$490 = 4\sqrt{A}$$

$$A^* = 15{,}006.25$$

From $P = 51 + \frac{\sqrt{A}}{10}$

$$P = 51 + \frac{\sqrt{15{,}006.25}}{10}$$

$$P^* = 63.25$$

<u>Ans</u> $A^* = 15{,}006.25$ and $P^* = 63.25$

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c) $H = \begin{bmatrix} \pi_{PP} & \pi_{PA} \\ \pi_{AP} & \pi_{AA} \end{bmatrix}$

$H = \begin{bmatrix} -40 & 2A^{-1/2} \\ 2A^{-1/2} & \frac{-p+2}{A\sqrt{A}} \end{bmatrix}$

$|H_1| = -40$ < 0

$$\left.\begin{aligned} |H_2| &= (-40)\left(\frac{-p+2}{A\sqrt{A}}\right) - (2A^{-1/2})^2 \\ &\text{from } A^* = 15,006.25 \text{ and } p^* = 63.25 \\ |H_2| &= (-40)\left(\frac{-63.25+2}{15,006.25\sqrt{15,006.25}}\right) - \left(\frac{2}{\sqrt{15,006.25}}\right)^2 \end{aligned}\right\} > 0$$

Hence, $|H|$ is negative definite and $d^2\pi < 0$ is local max

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d) From $\pi = (2000 + 4\sqrt{A} - 20p)(p-2) - 1000 - A$

$A^* = 15,006.25$ and $p^* = 63.25$

Then, $\pi = (2000 + 4\sqrt{15,006.25} - 20(63.25))(63.25-2) - 1000 - 15,006.25$

Ans $\pi = 59,025$

Question 3:

Consider a simple two-market model where demand and supply for each market is given below. (Notationally, let's name the two markets as A and B, respectively.)

Market A:

Demand: $p_A = 10 - 2Q_A$

Supply: $p_A = 1 + Q_A$

Market B:

Demand: $p_B = 20 - Q_B$

Supply: $p_B = 2 + 2Q_B$

a. Derive the market equilibrium
b. Suppose the government imposes unit tax on consumers in both markets at the rate of t_A and t_B. Solve for the after-tax equilibrium as the function of t_A and t_B.
c. How much revenue can the government collect from the taxation?
d. Determine the level of t_A and t_B that maximizes government's revenue.
e. Use the second-order conditions test and show that the answer obtained in "d" is a global solution.

a. Market A : $P_A = P_A$

$10 - 2Q_A = 1 + Q_A$

$9 = 3Q_A$

$Q_A = 3$

$P_A = 4$

Market B : $P_B = P_B$

$20 - Q_B = 2 + 2Q_B$

$18 = 3Q_B$

$Q_B = 6$

$P_B = 14$

b) T on consumer $P^d = P^s + t$

Market A

$P^d = 10 - 2Q$

$P^s = 1 + Q$

$P^* = 10 - 2Q = 1 + Q + t$

$Q_A = \frac{9-t}{3}$

$P^s = \frac{12-t}{3}$

$P^d = \frac{12+2t}{3}$

Market B

$P^d = P^s + t$

$P^d = 20 - Q$

$P^s = 2 + 2Q$

$P^* = 2 + 2Q + t = 20 - Q$

$Q = \frac{18-t}{3}$

$P^s = \frac{42-2t}{3}$

$P^d = \frac{42+t}{3}$

c) $TR = t_a \times Q_a + t_b \times Q_b$

$= t_a\left(\frac{9-t_a}{3}\right) + t_b\left(\frac{18-t_b}{3}\right)$

$= \frac{9t_a - t_a^2 + 18t_b - t_b^2}{3}$

d) $\frac{\partial TR}{\partial t_a} = 3 - \frac{2}{3}t_a = 0$

$t_a = 4.5$

$\frac{\partial TR}{\partial t_b} = 6 - \frac{2}{3}t_b = 0$

$t_b = 9$

E) $H = \begin{bmatrix} f_{aa} & f_{ab} \\ f_{ba} & f_{bb} \end{bmatrix} = \begin{bmatrix} -\frac{2}{3} & 0 \\ 0 & -\frac{2}{3} \end{bmatrix}$

$H_1 = -\frac{2}{3} < 0$

$H_2 = \frac{4}{9} > 0$

Hence Negative definite ; the function is global convex. Since any x, y yield the same result.