Question 1:

Given the equation for the production function

$$Q = f(K, L) = 18 * [0.2K^{-0.4} + 0.8L^{-0.4}]^{-2.5}$$

1.1 What type of constant return to scale does the production function exhibit?

1.2 Is the production function increasing with respect to K and L?

1.3 Use the implicit function rule to find the marginal rate of technical substitution (MRTS) of L for K.

1.4 Use the Hessian matrix. Proof that the production function is concave.

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1.1 $Q = 18[0.2k^{-0.4} + 0.8L^{-0.4}]^{-2.5}$

$$tQ = 18\left[(0.2)(kt)^{-0.4} + (0.8)(Lt)^{-0.4}\right)^{-2.5}$$

$$= 18[t^{-0.4\times-2.5}][0.2k + 0.8L]$$

$HM_0 = (-0.4)(-2.5) = 1$

∴ This production function has constant return to scale

1.2 $$\frac{\partial Q}{\partial k} = -45\left(0.2k^{-0.4} + 0.8L^{-0.4}\right)^{-3.5} \cdot \left(-0.08k^{-1.4}\right)$$

$$= \frac{-45}{(0.2k^{-0.4} + 0.8L^{-0.4})^{3.5}} \cdot \frac{-0.08}{k^{1.4}} = \frac{3.6}{(0.2k^{-0.4} + 0.8L^{-0.4})^{3.5} \cdot K^{1.4}} > 0$$

$$\frac{\partial Q}{\partial L} = -45\left(0.2k^{-0.4} + 0.8L^{-0.4}\right)^{-3.5} \cdot \left(-0.32L^{-1.4}\right)$$

$$= \frac{-45}{(0.2k^{-0.4} + 0.8L^{-0.4})^{3.5}} \cdot \frac{-0.32}{L^{1.4}} = \frac{14.4}{(0.2k^{-0.4} + 0.8L^{-0.4})^{3.5} \cdot L^{1.4}} > 0$$

∴ The production function will increase with respect to k, L

1.3 $f(K,L)=0$

$$f(K,L) = 18\left[0.2k^{-0.4} + 0.8L^{-0.4}\right]^{-2.5}$$

$$\frac{dy}{dx} = \frac{F_K}{F_L} = \frac{-45\left[0.2k^{-0.4} + 0.8L^{-0.4}\right]^{-3.5}\cdot\left[-0.08k^{-1.4}\right]}{-45\left[0.2k^{-0.4} + 0.8L^{-0.4}\right]^{-3.5}\cdot\left[-0.32L^{-1.4}\right]}$$

$$= \frac{-0.08k^{-1.4}}{-0.32L^{-1.4}} = \frac{0.32L^{1.4}}{0.08k^{1.4}} = 4\cdot\frac{L^{1.4}}{k^{1.4}}$$

$\therefore$ MRTS of L for k is $1/4$

1.4

From

$$\frac{\partial Q}{\partial k} = -45\left(0.2k^{-0.4} + 0.8L^{-0.4}\right)^{-3.5}\cdot\left(-0.08k^{-1.4}\right)$$

$$\frac{\partial Q}{\partial L} = -45\left(0.2k^{-0.4} + 0.8L^{-0.4}\right)^{-3.5}\cdot\left(-0.32L^{-1.4}\right)$$

Using Hessian Matrix

$$\frac{\partial^2 Q}{\partial KK} = -45\left(0.2k^{-0.4} + 0.8L^{-0.4}\right)^{-3.5}\cdot 0.112k^{-2.4} + \left(157.5\left(0.2k^{-0.4} + 0.8L^{-0.4}\right)^{-4.5} \times -0.08k^{-1.4}\right)\left(-0.08k^{-1.4}\right)$$

f_{kk}

$$\frac{\partial^2 Q}{\partial KL} = \frac{\partial^2 Q}{\partial LK} = -45\left(0.2k^{-0.4} + 0.8L^{-0.4}\right)^{-3.5} + \left(-0.08k^{-1.4}\right)\left(157.5\left(0.2k^{-0.4} + 0.8L^{-0.4}\right)^{-4.5} \times -0.32L^{-1.4}\right)$$

f_{kL} and f_{Lk}

$$\frac{\partial^2 Q}{\partial LL} = -45\left(0.2K^{-0.4} + 0.8L^{-0.4}\right)^{-3.5}\left(0.448L^{-2.4}\right) + \left(157.5\left(0.2K^{-0.4} + 0.8L^{-0.4}\right)^{-4.5} \times -0.32L^{-1.4}\right)\left(-0.32L^{-1.4}\right).$$

f_{LL}.

$$H = \begin{bmatrix} f_{KK} & f_{KL} \\ f_{LK} & f_{LL} \end{bmatrix}$$

$$|H_1| = -45\left(0.2K^{-0.4} + 0.8L^{-0.4}\right)^{-3.5} \cdot 0.112K^{-2.4} + \left(157.5\left(0.2K^{-0.4} + 0.8L^{-0.4}\right)^{-4.5}\left(0.0064K^{-2.8}\right)\right) < 0$$

$$|H_2| = \left[(f_{LK})(f_{KL}) - (f_{KK})(f_{LL})\right] > 0$$

$\therefore$ The production function is concave

Question 2: Define f(x,y) for all (x,y) by

$$f(x,y) = e^{x+y} + e^{x-y} - \frac{3}{2}x - \frac{1}{2}y$$

2.1 Derive the Hessian matrix of $f(x,y)$.

2.2 Show that f(x,y) is monotonically convex function. That's, the function is convex for the entire domain sets defining f(x,y).

2.3 Find the *global* extrema of f(x,y). What type of extrema is it?

2.1

$$f_x = \frac{\partial f}{\partial x} = e^{x+y} + e^{x-y} - \frac{3}{2}$$

$$f_y = \frac{\partial f}{\partial y} = e^{x+y} - e^{x-y} - \frac{1}{2}$$

$$f_{xx} = e^{x+y} + e^{x-y}$$

$$f_{yy} = e^{x+y} + e^{x-y}$$

$$f_{xy} = e^{x+y} - e^{x-y}$$

$$f_{yx} = e^{x+y} - e^{x-y}$$

$$H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$$

$$= \begin{bmatrix} e^{x+y} + e^{x-y} & e^{x+y} - e^{x-y} \\ e^{x+y} - e^{x-y} & e^{x+y} + e^{x-y} \end{bmatrix}$$

2.2 from 2.1 ;

$$|H_1| = e^{x+y} + e^{x-y} > 0 \quad \forall x, \forall y$$

$$|H_2| = (e^{x+y} + e^{x-y})(e^{x+y} + e^{x-y}) - (e^{x+y} - e^{x-y})(e^{x+y} - e^{x-y})$$

$$= (e^{x+y} + e^{x-y})^2 - (e^{x+y} - e^{x-y})^2 > 0 \quad \forall x, \forall y$$

$\therefore$ H is positive definite

$d^2f > 0 \quad \forall x, \forall y$

then f is monotonically convex function

2.3

$$\frac{\partial f}{\partial x} = e^{x+y} + e^{x-y} - \frac{3}{2} = 0 \quad \text{—}1$$

$$\frac{\partial f}{\partial x} = e^{x+y} - e^{x-y} - \frac{1}{2} = 0 \quad \text{—}2$$

1 + 2 ; $e^{x+y} + \cancel{e^{x-y}} + e^{x+y} - \cancel{e^{x-y}} - \frac{3}{2} - \frac{1}{2} = 0$

$$2e^{x+y} = \frac{4}{2}$$

$$e^{x+y} = 1$$

$$e^{x+y} = e^{0}$$

$$x+y = 0$$

$$x = -y$$

plug $x = -y$ in 1 ; $e^{-y+y} + e^{-y-y} - \frac{3}{2} = 0$

$$e^{0} + e^{-2y} = \frac{3}{2}$$

$$1 + e^{-2y} = \frac{3}{2}$$

$$e^{-2y} = \frac{3}{2} - 1$$

$$e^{-2y} = \frac{1}{2}$$

$$-2y = \ln \frac{1}{2}$$

$$-2y = \underbrace{\cancel{\ln 1}}_{0} - \ln 2$$

$$-2y = -\ln 2$$

$$y = \frac{\ln 2}{2}$$

$$\therefore \quad x = -\frac{\ln 2}{2}$$

$\therefore$ critical point is $\left(-\frac{\ln 2}{2}, \frac{\ln 2}{2}\right)$

At $\left(-\frac{\ln 2}{2}, \frac{\ln 2}{2}\right)$ is local minimizer because $f(x, y)$ is monotonically convex function, then $\left(-\frac{\ln 2}{2}, \frac{\ln 2}{2}\right)$ is the global minimizer.

- The global minimum is

$$f\left(-\frac{\ln 2}{2}, \frac{\ln 2}{2}\right)$$

$$= e^{-\frac{\ln 2}{2} + \frac{\ln 2}{2}} + e^{-\frac{\ln 2}{2} - \frac{\ln 2}{2}} + \frac{3}{2}\frac{\ln 2}{2} - \frac{1}{2}\frac{\ln 2}{2}$$

$$= e^{0} + e^{-\ln 2} + \frac{3}{4}\ln 2 - \frac{1}{4}\ln 2$$

$$= 1 + \frac{1}{e^{\ln 2}} + \frac{1}{2}\ln 2 = 1 + \frac{1}{2} + \frac{\ln 2}{2} = \frac{3}{2} + \frac{\ln 2}{2}$$

Question 3:

A monopolist faces the market demand given by $P = Q^{-c}$ where "c" is a parameter with positive value, "P" is the price per unit output and "Q" is the amount of output. Suppose that monopolist's production technology is given by $Q = K^{\frac{1}{3}}L^{\frac{2}{3}}$ where "K" and "L" are the level of capital used and the number of labor employed, respectively. Assume that the unit price of K and L are set equal to "r" and "w", respectively. Consider the following problems.

3.1) What type of the return to scale technology does the production function exhibit?

From now on, assume that c = ¼ . Consider the following problems.

Suppose we increase L & K by t

$$\text{Then, } (tK)^{1/3}(tL)^{2/3} = t^{1/3}\cdot t^{2/3}K^{1/3}L^{2/3} = tK^{1/3}L^{2/3} = tQ$$

∴ Degree of homogeneity is equal to 1, reflecting constant return to scale. #

3.2) Construct the profit function of the monopolist. (Hint: your profit function should be expressed in terms of K and L.)

$$TR = P\cdot Q = Q^{-c}\cdot Q = Q^{1-\frac{1}{4}} = (K^{1/3}L^{2/3})^{3/4}$$

$$TC = rK + wL$$

$$\text{Profit function} = TR - TC = (K^{1/3}L^{2/3})^{3/4} - (rK + wL) = (K^{1/3}L^{2/3})^{3/4} - rK - wL = K^{1/4}L^{1/2} - rK - wL \ *$$

3.3) The firm wants to maximize profit and seek for combination of the two factor inputs . Derive the demand for factor inputs, capital and labor.

$$\pi(K,L) = K^{1/4}L^{1/2} - rK - wL$$

$$\text{F.O.C.} \quad \frac{\partial \pi}{\partial K} = \frac{1}{4}K^{-3/4}L^{1/2} - r = 0$$

$$4rK^{3/4} = L^{1/2}$$

$$K = \left(\frac{L^{1/2}}{4r}\right)^{4/3} \quad \text{—— ①}$$

$$\frac{\partial \pi}{\partial L} = \frac{1}{2}K^{1/4}L^{-1/2} - w = 0$$

$$2wL^{1/2} = K^{1/4}$$

$$L = \left(\frac{K^{1/4}}{2w}\right)^{2} \quad \text{—— ②}$$

Substitute ② into ① ;

$$K = \left[\frac{\left[\left(\frac{K^{1/4}}{2w}\right)^{2}\right]^{1/2}}{4r}\right]^{4/3}$$

$$K = \left(\frac{K^{1/4}}{2w}\right)^{4/3}\cdot\left(\frac{1}{4r}\right)^{4/3}$$

$$K = \frac{K^{1/3}}{(8wr)^{4/3}}$$

$$K^{2/3} = \frac{1}{16w^{4/3}r^{4/3}}$$

$$K = \left(\frac{1}{16w^{4/3}r^{4/3}}\right)^{3/2}$$

$$K = \frac{1}{8^2w^2r^2}$$

$$K^{*} = \frac{1}{(8wr)^2}$$

$$L^* = \left[\frac{\left(\frac{1}{(8wr)^2}\right)^{1/4}}{2w} \right]^2$$

$$L^* = \left(\frac{1}{(8wr)^2}\right)^{1/2} \cdot \frac{1}{(2w)^2}$$

$$L^* = \frac{1}{(8wr)(4w^2)}$$

$$L^* = \frac{1}{32w^3r}$$

∴ The demand for factor inputs; Capital demand; $p \cdot MP_K = VMP_K = \frac{1}{(8wr)^2}$

Labor Demand; $p \cdot MP_L = VMP_L = \frac{1}{32w^3r}$

3.4) How does the demand for labor vary with respect to w and r? Show your result by using partial derivative.

$$L^* = \frac{1}{32w^3r}$$

$\frac{\partial L^*}{\partial w} = -\frac{3}{32w^4r}$; when the wage increases by 1 unit, the optimal amount of labor falls by $\frac{3}{32w^4r}$ units.

$\frac{\partial L^*}{\partial r} = -\frac{1}{32w^3r^2}$; when the price of capital increases by 1 unit, the optimal amount of labor falls by $\frac{1}{32w^3r^2}$ units.

3.5) Confirm your answer with the second-order condition.

$\pi_K = \frac{1}{4}K^{-3/4}L^{1/2} - r$; $\pi_L = \frac{1}{2}K^{1/4}L^{-1/2} - w$

$$H = \begin{bmatrix} \pi_{KK} & \pi_{KL} \\ \pi_{LK} & \pi_{LL} \end{bmatrix} = \begin{bmatrix} -\frac{3}{16}K^{-7/4}L^{1/2} & \frac{1}{8}K^{-3/4}L^{-1/2} \\ \frac{1}{8}K^{-3/4}L^{-1/2} & -\frac{1}{4}K^{1/4}L^{-3/2} \end{bmatrix}$$

$$|H_1| = \left| -\frac{3}{16}K^{-7/4}L^{1/2} \right| = -\frac{3}{16}K^{-7/4}L^{1/2}$$

$$|H_2| = \begin{vmatrix} -\frac{3}{16}K^{-7/4}L^{1/2} & \frac{1}{8}K^{-3/4}L^{-1/2} \\ \frac{1}{8}K^{-3/4}L^{1/2} & -\frac{1}{4}K^{1/4}L^{-3/2} \end{vmatrix} = \frac{1}{32}K^{-3/2}L^{-1}$$

plug in $K^* = \frac{1}{(8wr)^2}$, $L^* = \frac{1}{32w^3r}$ into $|H_1|, |H_2|$ $\rightarrow$ $|H_1| < 0$, $|H_2| > 0$ at (K^*, L^*)

∴ H is negative definite at K^*, L^* ; $d^2\pi < 0$ at K^*, L^*

Try any (K, L) other than (K^*, L^*) $\longrightarrow$ always $|H_1| < 0$, $|H_2| > 0$

∴ Always negative definite ; $d^2\pi$ is always less than 0

π is globally concave local maximizer is also a global maximizer.