Macroeconomics

Lecture 5

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Stochastic Control Problem

$$Max \quad E_0 \sum_{t=0}^{\infty} \beta^t U(x_t, c_t), \qquad 0 < \beta < 1 \qquad (17)$$

$$s.t. \quad x_{t+1} = g(x_t, c_t, \varepsilon_{t+1}), \qquad x_0 \ \ given \qquad (18)$$

ε_{t+1} *is a sequence of i.i.d.(independently and identically distributed) random variables.*

ε_{t+1} *is realized at* $t+1$ *after* c_t *has been decided at time* t.

Bellman's equation,

$$V(x) = \max_c \left[U(x,c) + \beta E\left[V\left[g\left(x,c,\tilde{\varepsilon}\right)\right] \mid x \right]\right] \qquad (19)$$

The 1st – order condition for the problem on the RHS of (19) is

$$\frac{\partial U(x,c)}{\partial c} + \beta E\left[\frac{\partial g\left(x,c,\tilde{\varepsilon}\right)}{\partial c} V'\left[g\left(x,c,\tilde{\varepsilon}\right)\right] \mid x \right] = 0 \qquad (20)$$

Stochastic Control Problem

The value function must also satisfies (this is analogous to eq. (9))

$$V'(x) = \frac{\partial U\left[x,h(x)\right]}{\partial x} + \frac{\partial U\left[x,h(x)\right]}{\partial h(x)} \cdot \frac{\partial h(x)}{\partial x}$$

$$+ \beta E\left\{ \frac{\partial g\left(x,h(x),\tilde{\varepsilon}\right)}{\partial x} V'\left(g\left(x,h(x),\tilde{\varepsilon}\right)\right) \mid x \right\} \qquad (21)$$

Consumption with a Random Return

$$\max \quad E_0 \sum_{t=0}^{\infty} \beta^t U\left(c_t\right), \qquad 0 < \beta < 1$$

$$s.t. \quad A_{t+1} = R_t\left(A_t - c_t\right),\ t \geq 0,\ A_0 \ given,$$

$$and \quad assumed \quad that \quad U'\left(c\right) > 0,\ U''\left(c\right) < 0.$$

$$\therefore Bellman's \ equation \ is$$

$$V\left(A_t, R_{t-1}\right) = \max_{s_t}\left\{U\left(A_t - s_t\right) + \beta E_t V\left(s_t R_t, R_t\right)\right\},$$

$$where \quad s_t = A_t - c_t,$$

$$The \ 1st-order \ condition \ \left(w.r.t. \ s_t\right)$$

$$-U'(c_t) + \beta E_t\left[\frac{\partial V\left(s_t R_t, R_t\right)}{\partial s_t R_t}.R_t \mid A_t, R_{t-1}\right] = 0. \qquad (22)$$

Consumption with a Random Return

Then from (21),

$$V'(A_t,R_{t-1})=\frac{\partial U\left[A_t-h(A_t,R_{t-1})\right]}{\partial c_t}.\frac{\partial c_t}{\partial A_t}$$

$$+\beta E\left\{\frac{\partial h(A_t,R_{t-1}).R_t}{\partial A_t}V'\left(h(A_t,R_{t-1}).R_t,R_t\right)\mid A_t,R_{t-1}\right\}$$

$$=\frac{\partial U[c_t]}{\partial c_t}\left[1-\frac{\partial h(A_t,R_{t-1})}{\partial A_t}\right]$$

$$+\frac{\partial h(A_t,R_{t-1})}{\partial A_t}\left[\beta E\left\{R_tV'\left(h(A_t,R_{t-1}).R_t,R_t\right)\mid A_t,R_{t-1}\right\}\right]$$

$$=\frac{\partial U(c_t)}{\partial c_t}-\frac{\partial h(A_t,R_{t-1})}{\partial A_t}\left[\frac{\partial U(c_t)}{\partial c_t}-\left[\beta E\left\{R_t.V'\left(h(A_t,R_{t-1}).R_t,R_t\right)\mid A_t,R_{t-1}\right\}\right]\right]$$

$$=\frac{\partial U(c_t)}{\partial c_t}.\quad (\because 1st-order\ condition\ w.r.t.\ s_t)$$

Hence, $E_t\{V'(A_{t+1},R_t)\}=E_t\{U'(c_{t+1})\}.$

Substituting the above equation into equation (22), *one has*

$$U'(c_t)=\beta E_t\left[U'(c_{t+1}).R_t\right] \qquad (23)$$

$\Rightarrow$ *an optimal saving policy function* $s_t=h(A_t,R_{t-1})$

$$Let \qquad U(c_t) = \ln c_t,$$

$$Assume\ that\ R_t\ be\ an\ i.i.d.\ random\ process\ such\ that\ \ 1 \le ER_t < (1/\beta^2)$$

$$If\ \ c_t = \gamma A_t,\ then\ (23)\ becomes$$

$$\frac{1}{\gamma A_t} = \beta E_t\left[\frac{R_t}{\gamma A_{t+1}}\right] = \beta E_t\left[\frac{R_t}{\gamma R_t\left(A_t - \gamma A_t\right)}\right], \quad \left(\because A_{t+1} = R_t\left(A_t - c_t\right)\right)$$

$$Hence, \quad \frac{1}{\gamma A_t} = \beta E\left[\frac{1}{\gamma\left(A_t - \gamma A_t\right)}\right],$$

$$\gamma\left(A_t - \gamma A_t\right) = \beta\gamma A_t,$$

$$\left(1-\beta\right)\gamma A_t = \gamma^2 A_t,$$

$$\therefore \qquad \gamma = 1-\beta,$$

$$Then, \qquad c_t = \left(1-\beta\right)A_t.$$

$$Note\ that \qquad A_1 = R_0\left[A_0 - c_0\right]$$

$$\therefore \qquad A_1 = R_0\left(1-\gamma\right)A_0, \quad \left(\because c_t = \gamma A_t\right)$$

$$A_2 = R_1\left(\beta\right)A_1 = R_1\beta R_0\left(1-\gamma\right)A_0 = \beta^2 R_1 R_0 A_0$$

$$\vdots$$

$$A_t = \beta^t\left[\prod_{j=0}^{t-1} R_j\right]A_0, \qquad t = 1,2,.....$$

$$\therefore \quad c_t = \gamma A_t = \left(1-\beta\right)\beta^t\left[\prod_{j=0}^{t-1} R_j\right]A_0, \qquad t = 1,2,.....$$