Curve Sketching: Computing Limits for Finding Asymptotes

1 Review: Limit Definition

1.1 One-sided limits & Two-sided limits (normal limits)

As the name implies, with one-sided limits we will only be looking at one side of the point in question. Here are the definitions for the two one sided limits.

Definition 1.1. (Limits)
(I) **One-sided limits**
- **Right-handed limit:** We say

$$\lim_{x\to a^+} f(x) = L_1 \tag{1}$$

provided that we can make $f(x)$ as close to L_1 as we want for all x sufficiently close to a and $x > a$ without actually letting x be a.
- **Left-handed limit:** We say

$$\lim_{x\to a^-} f(x) = L_2 \tag{2}$$

provided that we can make $f(x)$ as close to L_2 as we want for all x sufficiently close to a and $x < a$ without actually letting x be a.
(II) **Two-sided limits (normal limits)** Given a function f(x), the normal limit of $f(x)$ exists and equal to L as x approaches a, i.e.

$$\lim_{x\to a} f(x) = L$$

if and only if

$$\lim_{x\to a^+} f(x) = L \qquad \text{and} \qquad \lim_{\text{x}\to \text{a}^-} \text{f(x)} = \text{L}.$$

Remarks

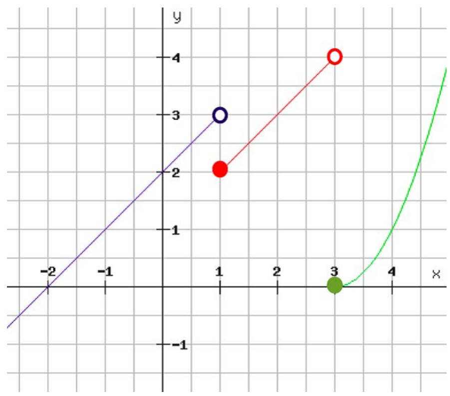
- For the right-handed limit we now have (note the "+") which means that we know will only look at $x > a$. Likewise for the left-handed limit we have (note the "-") which means that we will only be looking at $x < a$.
- One-sided limits don't have to exist just as normal limits are not guaranteed to exist.
- In general, the (two-sided) limit $\lim_{x\to a} f(x)$ does not exist when
  - either of one-sided limits $\lim_{x\to a^-} f(x)$ or $\lim_{x\to a^+} f(x)$ does not exist;
  - both two one-sided limits exit, but they have different values, i.e.,

$$\lim_{x\to a^-} f(x) \neq \lim_{x\to a^+} f(x).$$

- **Existence and Nonexistence:** *The existence of a limit of a function f as x approaches a (from one side of from both sides) does not depend on whether f defined at a but only on whether f is defined for x near the number a.*

Example 1.1. Let $f(x) = \begin{cases} x+2, & \text{for} \quad x<1 \\ x+1, & \text{for} \quad 1 \le x < 3 \\ (x-3)^2, & \text{for} \quad x \ge 3 \end{cases}$.

Plot the function f and find the following limits, or state that it does not exist.



(a) $\lim_{x\to 1^-} f(x) =$ $\qquad$ $\lim_{x\to 1^+} f(x) =$

$\lim_{x\to 1} f(x)$ does not exist because left-hand limit and right-hand limit are different.

(b) $\lim_{x\to 3^-} f(x) =$ $\qquad$ $\lim_{x\to 3^+} f(x) =$

$\lim_{x\to 3} f(x)$ does not exist because left-hand limit and right-hand limit are different.

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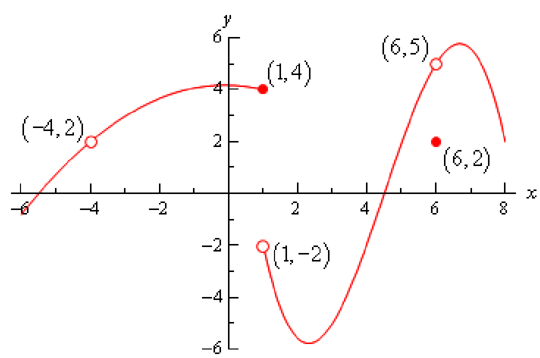
Example 1.2. Consider the following plot of a function $f(x)$. Find the following limits, or state that it does not exist.



(a) $\lim_{x\to -4^-} f(x)=$ $\qquad$ $\lim_{x\to -4^+} f(x) =$

$\lim_{x\to -4} f(x) =$

(b) $\lim_{x\to 1^-} f(x)=$ $\qquad$ $\lim_{x\to 1^+} f(x) =$

$\lim_{x\to 1} f(x) =$

(c) $\lim_{x\to 6^-} f(x)=$ $\qquad$ $\lim_{x\to 6^+} f(x) =$

$\lim_{x\to 6} f(x) =$

Theorem 1.1. Assume that $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist. Let c be any constant. Then,

1. $\lim_{x \to a} c\, f(x) = c \lim_{x \to a} f(x)$, and $\lim_{x \to a} c = c$
2. $\lim_{x \to a} [\, f(x) \pm g(x)] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$,
3. $\lim_{x \to a} [f(x)g(x)] = \left[\lim_{x \to a} f(x)\right] \left[\lim_{x \to a} g(x)\right]$,
4. $\lim_{x \to a} \left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$, provided that $\lim_{x \to a} g(x) \neq 0$,
5. $\lim_{x \to a} [f(x)]^k = \left[\lim_{x \to a} f(x)\right]^k$, where k is a constant.

Limit of a polynomial function

Let $f(x)$ be a polynomial function of degree n, i.e. $f(x)$ is in the form

$$f(x) = c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0,$$

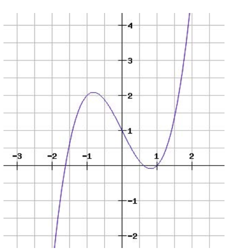
where $c_0, c_1, \ldots c_n$ are constant. Then, for any constant real number a,

$$\lim_{x \to a^-} f(x) = f(a) \qquad \lim_{x \to a^+} f(x) = f(a) \qquad \lim_{x \to a} f(x) = f(a) \tag{3}$$

Example 1.3. Consider the function

$$f(x) = x^3 - 2x + 1$$

Find the following limits: $\lim_{x \to 0^-} f(x) =$ $\lim_{x \to 0^+} f(x) =$ and $\lim_{x \to 0} f(x) =$.



Example 1.4. Evaluate the following limits.

$\lim_{x \to -1} x^5 - x^3 + 3x^2 - 1$

$\lim_{x \to 1} x^5 - 2x^2 + 1$

$\lim_{x \to -1} (2x + 1)^8$

2 Vertical Asymptote

By using the equation in the form $y = f(x)$, $\boxed{x = a}$ is a vertical asymptote if

$$\boxed{\begin{array}{l} \lim_{x\to a} f(x) = \infty \quad \text{or} \quad \lim_{x\to a^+} f(x) = \infty \quad \text{or} \quad \lim_{x\to a^-} f(x) = \infty \quad \text{or} \\ \lim_{x\to a} f(x) = -\infty \quad \text{or} \quad \lim_{x\to a^+} f(x) = -\infty \quad \text{or} \quad \lim_{x\to a^-} f(x) = -\infty. \end{array}}$$

Example 2.1. Each of the following plots of function has $\boxed{x = 1}$ as a **vertical asymptote**.

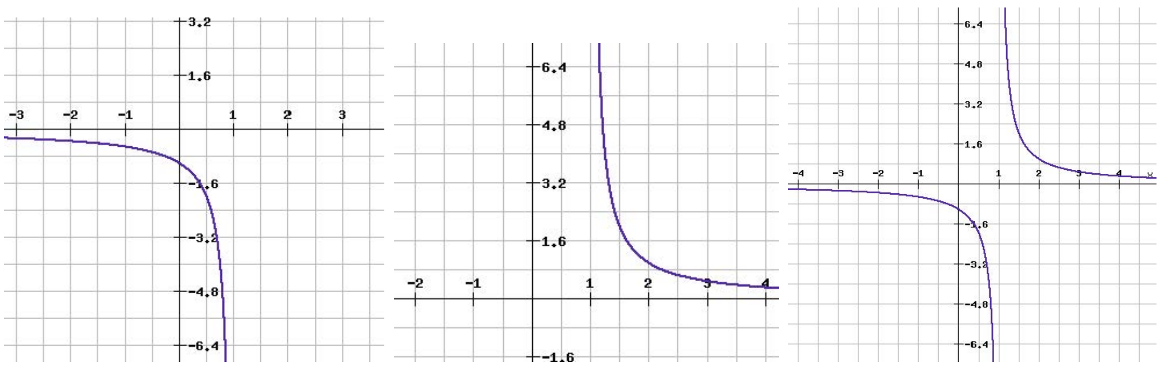


Figure 1: (i) $\lim_{x\to a^-} f(x) = -\infty$, (ii) $\lim_{x\to a^+} f(x) = \infty$ and (iii) $\lim_{x\to a^-} f(x) = -\infty$ $\lim_{x\to a^+} f(x) = \infty$, for $a = 1$.

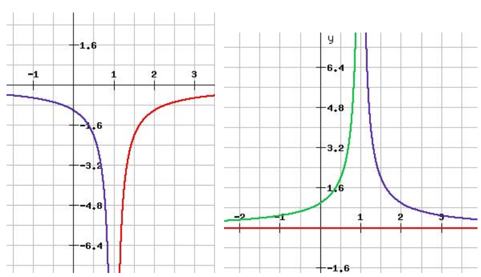


Figure 2: Left: $\lim_{x\to a} f(x) = -\infty$; Right: $\lim_{x\to a} f(x) = -\infty$ where $a = 1$.

Consider a general rational function, which is in the form

$$f(x) = \frac{p(x)}{q(x)}$$

where p and q have no common factors. Then, if q contains a factor of the form $(x-a)^n$ where n is a positive integer, then $\boxed{x = a \text{ is a vertical asymptote}}$, since at least one of the following limits in the definition of vertical asymptotes is true, e.g. $\lim_{x\to a^-} \frac{p(x)}{q(x)} = \pm\infty$, or $\lim_{x\to a^+} \frac{p(x)}{q(x)} = \pm\infty$. Note that, in this case, $p(a) \neq 0$.

Remark: Computing $\lim_{x\to a}\frac{f(x)}{g(x)}$

Let f and g be functions and suppose $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist with

$$\lim_{x\to a} f(x) = L_1, \qquad \lim_{x\to a} g(x) = L_2.$$

We consider 3 cases of $\lim_{x\to a}\frac{f(x)}{g(x)}$:

(I) If $L_2 \neq 0$, $\lim\limits_{x\to a}\frac{f(x)}{g(x)} = \frac{L_1}{L_2}$.

(II) If $L_1 \neq 0$ and $L_2 = 0$, $\lim\limits_{x\to a}\frac{f(x)}{g(x)}$ **does not exist**.

In this case, $\lim\limits_{x\to a}\frac{f(x)}{g(x)} = \begin{cases} \infty & , \; L_1 > 0 \\ -\infty & , \; L_1 < 0 \end{cases}$. $\Longrightarrow$ *"Infinity Limits"*

(III) If $L_1 = 0$ and $L_2 = 0$, $\Longrightarrow$ *" Indeterminate Form $\frac{0}{0}$ "*

To find $\lim\limits_{x\to a}\frac{f(x)}{g(x)}$, we have to rearrange $\frac{f(x)}{g(x)}$, e.g.

- factor $f(x)$, $g(x)$ and eliminate the common terms before taking limit;
- use *rationalization*: multiply the same factor to $f(x)$ and $g(x)$ to use $(p-q)(p+q) = p^2 - q^2$.

Example 2.2. Determine the following limits.

(a) $\lim\limits_{x\to 0}\frac{3}{x}$ [Ans: ∞]

(b) $\lim\limits_{x\to 3}\frac{x}{3-x}$ [Ans: $-\infty$]

(c) $\lim\limits_{x\to -1}\frac{x^2+2x+1}{x^2-4x-5}$ [Ans: 0]

(d) $\lim\limits_{x\to 0}\frac{\sqrt{x^2+4}-2}{x^2}$ (Exercise) [Ans: 1/4]

[So, $x = 0$ is not a **vertical asymptote**]

Example 2.3. Find all vertical asymptotes of the following functions.

(a) $f(x) = \frac{3-x}{x}$

(b) $f(x) = \frac{x}{(3-x)(2x-1)}$

(c) $f(x) = \frac{x^2+2x+1}{x^2-4x-5}$

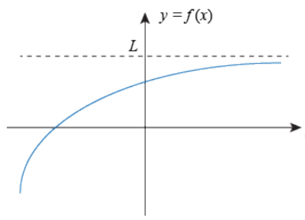
(d) $f(x) = \frac{x-3}{x^2-6x+9}$

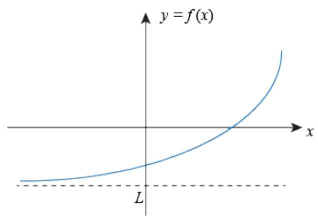
3 Horizontal asymptote

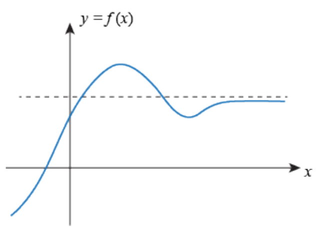
By using the equation in the form $y = f(x)$, $\boxed{y = L}$ is a **horizontal asymptote** if

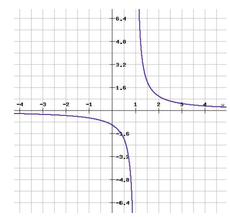
$$\boxed{\lim_{x\to\infty} f(x) = L \quad \text{or} \quad \lim_{x\to-\infty} f(x) = L.}$$

Alternatively, if it is possible to re-arrange $x = g(y)$, $y = L$ is a horizontal asymptote when
$\lim_{y\to L} g(y) = \infty$ or $\lim_{y\to L^+} g(y) = \infty$ or $\lim_{y\to L^-} g(y) = \infty$ or
$\lim_{y\to L} g(y) = -\infty$ or $\lim_{y\to L^+} g(y) = -\infty$ or $\lim_{y\to L^-} g(y) = -\infty$.









Theorem 3.1. [Limits at Infinity] If r is a positive rational number and a is any real number then,

$$\lim_{x\to\infty} \frac{1}{(x-a)^r} = 0 \qquad \text{and} \qquad \lim_{x\to-\infty} \frac{1}{(x-a)^r} = 0.$$

Example 3.1. From above theorem, we have $\lim_{x\to\infty} \frac{1}{x-1} = 0$ and $\lim_{x\to-\infty} \frac{1}{x-1} = 0$, which implies that $\boxed{y = 0}$ is a horizontal asymptote as shown in the figure above.

Example 3.2. Determine the horizontal asymptotes for the following functions.

- $f(x) = \frac{x}{x^2+1} + 1$
- $f(x) = \frac{2}{x(1-x)}$

Limits at infinity of rational functions

We will consider the limit of a function that can be written as a fraction of 2 functions $p(x)$ and $q(x)$. The following is what we have so far.
Recall that **rational** functions are in the form

$$f(x) = \frac{p(x)}{q(x)}$$

where $p(x)$ and $q(x) \neq 0$ are **polynomials**. We can compute limit at infinity by looking at the **end behavior** of $f(x)$ as $|x|$ gets large, which will reflect in the **highest degree** of each polynomial.

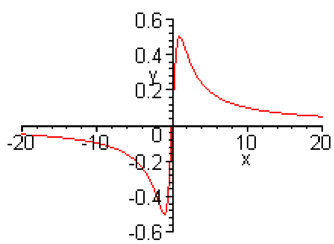
Theorem 3.2. Let $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$ be a polynomial of degree $n \geq 0$ (i.e. $a_n \neq 0$), and let $q(x) = b_n x^m + b_{m-1}x^{m-1} + \cdots + b_2 x^2 + b_1 x + b_0$ be a polynomial of degree $m \geq 0$ (i.e. $b_m \neq 0$). Then, for the **rational** function $f(x) = \frac{p(x)}{q(x)}$

$$\lim_{x\to\infty} \frac{p(x)}{q(x)} = \lim_{x\to\infty} \left(\frac{a_n x^n}{b_m x^m}\right) = \left(\frac{a_n}{b_m}\right) \lim_{x\to\infty} x^{n-m}$$

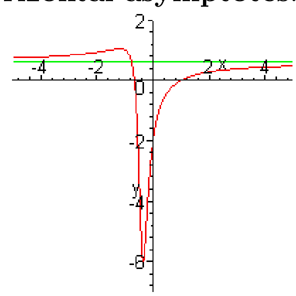
$$\lim_{x\to-\infty} \frac{p(x)}{q(x)} = \lim_{x\to-\infty} \left(\frac{a_n x^n}{b_m x^m}\right) = \left(\frac{a_n}{b_m}\right) \lim_{x\to-\infty} x^{n-m}.$$

That is, when we want to take a limit at infinity for a polynomial then all we need to really do is look at the term with the largest power and ask what that term is doing in the limit since the polynomial will have the same behavior.

Example 3.3. Evaluate the following limits and determine the **horizontal asymptotes**.



(a) $\frac{x}{1+x^2}$



(b) $\frac{3x^2-x-2}{5x^2+4x+1}$

(a) $\displaystyle\lim_{x\to-\infty} \frac{x}{x^2+1} =$ $\qquad$ $\displaystyle\lim_{x\to\infty} \frac{x}{x^2+1} =$

Horizontal asymptote(s):

(b) $\displaystyle\lim_{x\to-\infty} \frac{3x^2-x-2}{5x^2+4x+1} =$ $\qquad$ $\displaystyle\lim_{x\to\infty} \frac{3x^2-x-2}{5x^2+4x+1} =$

Horizontal asymptote(s):

Example 3.4. Evaluate the following limits and determine the **horizontal asymptotes**.

(a) $\displaystyle\lim_{x\to-\infty} \frac{1+2x^2-3x^3}{2x^3+3x} =$ $\qquad$ $\displaystyle\lim_{x\to\infty} \frac{1+2x^2-3x^3}{2x^3+3x} =$

Horizontal asymptote(s):

(b) $\displaystyle\lim_{x\to-\infty} \frac{x^5-3x^3-7}{4x^5-x} =$ $\qquad$ $\displaystyle\lim_{x\to\infty} \frac{x^5-3x^3-7}{4x^5-x} =$

Horizontal asymptote(s):

$(c)\ \lim_{x\to-\infty} \dfrac{x^5 - x^2 - 4}{x^7 - 2} =$ $\qquad \lim_{x\to\infty} \dfrac{x^5 - x^2 - 4}{x^7 - 2} =$

Horizontal asymptote(s):

$(d)\ \lim_{x\to-\infty} \dfrac{x^7 - 2}{x^5 - x^2 - 4} =$ $\qquad \lim_{x\to\infty} \dfrac{x^7 - 2}{x^5 - x^2 - 4} =$

Horizontal asymptote(s):

Exercise

1. Determine horizontal asymptotes (if any) of the following functions.

   (a) $f(x) = \frac{1+2x^2-3|x|^3}{2x^3+3x}$ [Ans: $y = 3/2, -3/2$]

   (b) $f(x) = \frac{x}{\sqrt{2+x^2}}$

   (c) $f(x) = \sqrt{\frac{x^5-3x^3-7}{4x^5-x}}$

2. Find all vertical asymptotes (if any) of the following functions.

   (a) $f(x) = \frac{2(x-3)^2-18}{x}$.

   (b) $f(x) = \begin{cases} \frac{x^2+3x+2}{x+1} & , \quad x \neq -1 \\ 3 & , \quad x = -1 \end{cases}$.

   (c) $f(x) = \begin{cases} \frac{x^2-1}{x-1} & , \quad x < 1 \\ x^2 & , \quad x \geq 1 \end{cases}$.

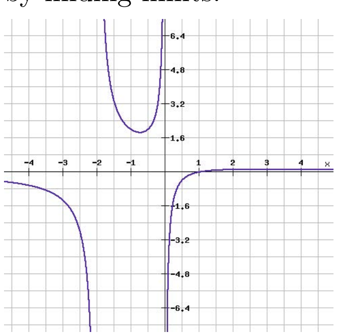
   (d) $f(x) = \frac{|x-2|}{x-2}$

   (e) $f(x) = \frac{x-1}{\sqrt{3-x}}$

3. Determine all vertical and horizontal asymptotes of the rational function

$$f(x) = \frac{x-1}{x^2+2x}$$

   by finding limits.



4 Review: Derivatives

Definition 4.1. The **derivative of** $f(x)$ **with respect to** x , denoted by $f'(x)$, is defined as

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}, \tag{4}$$

whenever the limit exists.

4.1 The Derivative: Power and Sum Rules

Theorem 4.1. Properties of Derivatives.

- [Constant Multiple Rule] If c is a constant and f is a differentiable function at x, then cf is a differentiable function at x and then $\frac{d}{dx}cf(x) = c\frac{df(x)}{dx} = cf'(x)$.
- [Sum and Difference Rules] Let f and g be differentiable functions at x. Then $f+g$ and $f-g$ are differentiable at x and

$$(f(x)+g(x))' = f'(x)+g'(x)$$

$$(f(x)-g(x))' = f'(x)-g'(x).$$

Theorem 4.2. [Power Rule] For any real number n,

$$\frac{d}{dx}x^n = nx^{n-1}.$$

If $f(x) = c$ is a constant function, then $f'(x) = 0$.

Example 4.1. Differentiate
(a) $y = x^{10} + 2x\sqrt{x^7} - e^{\pi}$

(b) $y = -3\sqrt[3]{x} + \frac{7}{x} + \frac{4}{x\sqrt{x}} + 9$

(c) $y = \frac{x^5 - x^{3/2} + 1}{\sqrt{x}}$

(d) $y = \frac{x^2+3x+2}{x+1}$
.

4.2 The Derivative: Product and Quotient Rules

Theorem 4.3 (Product Rule)**.** Let f and g be differentiable functions at x. Then the product fg is differentiable at x and

$$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}g(x) + g(x)\frac{d}{dx}f(x).$$

Theorem 4.4 (Quotient Rule)**.** Let f and g be *differentiable* functions at x and $g(x) \neq 0$. Then f/g is differentiable at x and

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}.$$

Example 4.2. Differentiate $y = (x^2 + 3x + 2)(x^3 - x + 1)$.

Example 4.3. (Exercise) Differentiate $y = (2 + x)(x^3 + 1)(x^2 - 7x)$.

Example 4.4. Differentiate

$$y = \frac{2 + x\sqrt{x}}{x^3 + 1}.$$

Example 4.5. Differentiate

$$y = \frac{x - 2}{(x + 1)(x^2 - 7x)}.$$

4.3 Chain Rule

Theorem 4.5. (Chain Rule): Let $f(x)$ and $g(x)$ be differentiable functions. Consider the composite function

$$F(x) = (f \circ g)(x) = f(g(x)).$$

Then the function $F(x)$ is differentiable and

$$\boxed{F'(x) = \frac{d}{dx}[f(g(x))] = f'(g(x))g'(x)}. \tag{5}$$

Alternatively,

$$\boxed{y = f(u) \text{ and } u = g(x) \quad \Longrightarrow \quad \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}}. \tag{6}$$

Note that $y = f(u) = f(g(x)) = F(x)$ and therefore the derivative of $F(x)$: $\frac{d}{dx}F(x) = \frac{dy}{dx}$.

Special Case: Power Rule for Functions

Let n be any real number. Suppose $u = g(x)$ is differentiable at x. Then

$$\boxed{\frac{d}{dx}[g(x)]^n = n[g(x)]^{n-1}\frac{d}{dx}g(x)}. \tag{7}$$

Equivalent,

$$\boxed{\frac{d}{dx}u^n = nu^{n-1}\frac{du}{dx}}. \tag{8}$$

Example 4.6. Differentiate $F(x) = (1 + 3x + 5x^5 + 4x^{10})^7$.

Definition 4.2. The n**-th** derivative of a function f is defined by taking derivative of the $(n-1)$**-th** derivative $f^{(n-1)}(x)$, for $n = 1, 2, 3, \ldots$. I.e., the n-th derivative of $y = f(x)$ with respect to x is $\boxed{\frac{d^n y}{dx^n} = \frac{d}{dx}\left(\frac{d^{(n-1)}y}{dx^{(n-1)}}\right).}$ Notation: $f^{(n)}(x) \quad \frac{d^n y}{dx^n} \quad \frac{d^n}{dx^n}f(x) \quad y^{(n)} \quad D_x^n \quad D^n$.

The second derivative of $y = f(x)$ is $f''(x) = \frac{d}{dx}\left(\frac{dy}{dx}\right)$. Other notations for the second derivative of $y = f(x)$ with respect to x are $f''(x) \quad \frac{d^2y}{dx^2} \quad \frac{d^2}{dx^2}f(x) \quad y'' \quad D_x^2 \quad D^2$.

Formulas of Derivatives

Suppose $u = g(x)$ is differentiable at x.

Derivatives of Exponential and Logarithmic Functions

$\frac{d}{dx}e^u = e^u \frac{du}{dx}$ $\qquad$ $\frac{d}{dx}b^u = b^u \ \ln(b)$

$\frac{d}{dx}\ln(|u|) = \frac{1}{u}\ \frac{du}{dx}$ $\qquad$ $\frac{d}{dx}\log_b(u) = \frac{1}{u\ \ln(b)}\ \frac{du}{dx}$

Derivatives of Trigonometric Functions & Their Inverse Functions:

$\frac{d}{dx}\sin(u) = \cos(u)\frac{du}{dx}$ $\qquad$ $\frac{d}{dx}\cos(u) = -\sin(u)\frac{du}{dx}$

$\frac{d}{dx}\tan(u) = \sec^2(u)\frac{du}{dx}$ $\qquad$ $\frac{d}{dx}\cot(u) = -\csc^2(u)\frac{du}{dx}$

$\frac{d}{dx}\sec(u) = \sec(u)\tan(u)\frac{du}{dx}$ $\qquad$ $\frac{d}{dx}\csc(u) = -\csc(u)\cot(u)\frac{du}{dx}.$

$\frac{d}{dx}\sin^{-1}(u) = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$ $\qquad$ $\frac{d}{dx}\cos^{-1}(u) = \frac{-1}{\sqrt{1-u^2}}\frac{du}{dx}, \quad |u| < 1$

$\frac{d}{dx}\tan^{-1}(u) = \frac{1}{1+u^2}\frac{du}{dx}$ $\qquad$ $\frac{d}{dx}\cot^{-1}(u) = \frac{-1}{1+u^2}\frac{du}{dx}$

$\frac{d}{dx}\sec^{-1}(u) = \frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx}$ $\qquad$ $\frac{d}{dx}\csc^{-1}(u) = \frac{-1}{|u|\sqrt{u^2-1}}\frac{du}{dx}, \quad |u| > 1.$

Derivatives of Hyperbolic Functions & Theirs Inverse Functions

$\frac{d}{dx}\sinh(u) = \cosh(u)\frac{du}{dx}$ $\qquad$ $\frac{d}{dx}\cosh(u) = \sinh(u) = \frac{du}{dx}$

$\frac{d}{dx}\tanh(u) = \text{sech}^2\frac{du}{dx}$ $\qquad$ $\frac{d}{dx}\coth(u) = -csch^2(u)\frac{du}{dx}$

$\frac{d}{dx}\text{sech}(u) = \text{sech}(u)\tanh(u)\frac{du}{dx}$ $\qquad$ $\frac{d}{dx}\text{csch}(u) = -\text{csch}(u)\coth(u)\frac{du}{dx}$

$\frac{d}{dx}\sinh^{-1}(u) = \frac{1}{\sqrt{u^2+1}}\frac{du}{dx}$ $\qquad$ $\frac{d}{dx}\cosh^{-1}(u) = \frac{-1}{\sqrt{u^2-1}}\frac{du}{dx}, \quad u > 1$

$\frac{d}{dx}\tanh^{-1}(u) = \frac{1}{1-u^2}\frac{du}{dx}, \quad |u| < 1$ $\qquad$ $\frac{d}{dx}\coth^{-1}(u) = \frac{1}{1-u^2}\frac{du}{dx}, \quad |u| > 1$

$\frac{d}{dx}\text{sech}^{-1}(u) = \frac{-1}{u\sqrt{1-u^2}}\frac{du}{dx}, \ 0 < u < 1$ $\qquad$ $\frac{d}{dx}\text{csch}^{-1}(u) = \frac{-1}{|u|\sqrt{1+u^2}}\frac{du}{dx}, \quad u \neq 0.$