Quiz 2: Date: May 5, 2022 from 11.00-12.30

Question 1 (40 marks)

Score.....................

Consider the Muliperiod model of consumption and portfolio choice. Let an individual in this economy has the utility function as follow:

$$\max_{C_s, \omega_s, \forall t} E_t\left[\sum_{s=t}^{T-1} \delta^s \left(\frac{C_s^{(1-\gamma)}}{1-\gamma}\right) + \delta^T \left(\frac{W_T^{(1-\gamma)}}{1-\gamma}\right)\right]$$

Assume that there is no wage income $(y_t = 0\, \forall t)$ and a constant risk-free rate return asset , $R_{ft} = R_f$. Also asssume that n=1 and the return of a single risky asset, R_{rt},is independently and identically distributed over time. Denote the proportion of wealth invested in the risky asset at date t as ω_t.

Please read and answer the following questions carefully and completely.

$$\max_{C_s,\omega_s,\forall_t} E_t\left[\sum_{s=t}^{T-1}\delta^s\left(\frac{C_s^{(1-\gamma)}}{1-\gamma}\right)+\delta^T\left(\frac{W_T^{(1-\gamma)}}{1-\gamma}\right)\right]$$

Score.....................

Question 1.1 (10 marks) Derive the first-order condition for the optimal consumption level and portfolio weight at date T-1, C^*_{T-1} and ω^*_{T-1}., and give an explicit expression for C^*_{T-1}

Note : $S_t = W_t - C_t$, $W_{t+1} = S_t R_t$

$$V_c(C^*_{T-1}, T-1) = E_{T-1}[B_W(W_T, T) R_{T-1}$$

$$\delta^{T-1}\frac{C_{T-1}^{(1-\gamma)}}{\cancel{1-\gamma}} = E_{T-1}\left[\delta^T\frac{W_T^{(1-\gamma)}}{1-\gamma}\cdot R_{T-1}\right] = E_{T-1}\left[\delta^T\left(\frac{S_{T-1}R_{T-1}}{\cancel{1-\gamma}}\right)^{1-\gamma}\cdot R_{T-1}\right]$$

$$= \delta^T E_{T-1}[R_{T-1}^{2-\gamma}](W_{T-1} - C_{T-1})^{1-\gamma}$$

or

$$C^*_{T-1} = \frac{(\delta E_{T-1}[R_{T-1}^{2-\gamma}])^{\frac{1}{1-\gamma}}}{1+(\delta E_{T-1}[R_{T-1}^{2-\gamma}])^{\frac{1}{1-\gamma}}}\cdot W_{T-1}$$

✗

F.o.c. wrt. portfolio weight

$$E_{T-1}[B_W R_{r,T-1}] = R_f E_{T-1}[B_W]$$

$$\delta^T E_{T-1}[(S_{T-1}R_{T-1})^{1-\gamma} R_{r,T-1}] = \delta^T R_{f,T-1} E_{T-1}[(S_{T-1}R_{T-1})^{1-\gamma}]$$

$$E[R_{T-1}^{1-\gamma} R_{r,T-1}] = R_{f,T-1} E[R_{T-1}^{1-\gamma}]$$

✗

$$c^{*}_{T-1} = \frac{\left(\delta E_{T-1}\left[R_{T-1}^{2-\gamma}\right]\right)^{\frac{1}{1-\gamma}}}{1+\left(\delta E_{T-1}\left[R_{T-1}^{2-\gamma}\right]\right)^{\frac{1}{1-\gamma}}} \cdot W_{T-1} = \frac{x_1}{1+x_1} W_{T-1}$$

Score......................

where $x_1 = \left(\delta E_{T-1}\left[R_{T-1}^{2-\gamma}\right]\right)^{\frac{1}{1-\gamma}}$

Question 1.2 (10 marks) Solve for the form of J(W_{T-1},T-1).

$$J(W_{T-1}, T-1) = \max_{C_{T-1},\{w_{T-1}\}} U(c_{T-1}, T-1) + E_{T-1}[B(W_T, T)]$$

$$J(W_{T-1}, T-1) = \delta^{T-1} \frac{c_{T-1}^{1-\gamma}}{1-\gamma} + \delta^{T} E_{T-1}\left[\frac{\left(R_{T-1}(W_{T-1} - c^{*}_{T-1})\right)^{1-\gamma}}{1-\gamma}\right]$$

$$= \delta^{T-1} \frac{\left(\frac{x_1}{1+x_1}\right)^{1-\gamma} \cdot W_{T-1}}{1-\gamma} + \delta^{T} E_{T-1}\left[R_{T-1}^{1-\gamma} \cdot \frac{W_{T-1}^{1-\gamma}}{1-\gamma(1+x_1)^{1-\gamma}}\right]$$

Score.....................

Question 1.3 (10 marks) Derive the first-order condition for the optimal consumption level and portfolio weight at date T-2, C^*_{T-2} and ω^*_{T-2}., and give an explicit expression for C^*_{T-2}

$$V_c(C^*_{T-2}, T-2) = E_{T-2}[J_W(W_{T-1}, T-1)R_{T-2}]$$

$$C^{\gamma}_{T-2} = \delta^{T-1} E_{T-2}[b_1 W^{\gamma}_{T-1} R_{T-2}]$$

$$C^{\gamma}_{T-2} = \delta E_{T-2}[b_1 (S_{T-2} R_{T-2})^{\gamma} R_{T-2}]$$

$$= \delta b_1 E_{T-2}[R^{\gamma-1}_{T-2}](W_{T-2} - C_{T-2})^{\gamma-1}$$

Score.....................

Question 1.4 (10 marks) Solve for the form of J(W_{T-2},T-2).Based on the pattern for T-1 and T-2, provide expressions for the optimal consumption and portfolio weight at any date T-t, t=1,2,3,....

$J(W_{T-2}, T-2) = V(C^*_{T-2}, T-2) + E_{T-1}[J(W_{T-1}, T-1)]$