Homework#1

Solution for question 17

Suppose that the real estimators for X_1, X_2 ,and X_3 are μ

17.1

$$E(\tilde{X}) = E\left(\frac{1}{3}X_1 + \frac{2}{3}X_2\right) = \frac{1}{3}E(X_1) + \frac{2}{3}E(X_2) = \frac{1}{3}\mu + \frac{2}{3}\mu = \mu$$

Hence, $\tilde{X}$ is an unbiased estimator.

17.2

$$E(\overline{X}) = E\left(\frac{X_1 + X_2 + X_3}{3}\right) = \frac{1}{3}\left(E(X_1) + E(X_2) + E(X_3)\right) = \frac{\mu + \mu + \mu}{3} = \mu$$

Hence, $\overline{X}$ is an unbiased estimator.

17.3

$$Var(\tilde{X}) = Var\left(\frac{1}{3}X_1 + \frac{2}{3}X_2\right) = \left(\frac{1}{3}\right)^2 Var(X_1) + \left(\frac{2}{3}\right)^2 Var(X_2) = \frac{5}{9}\sigma^2$$

$$Var(\overline{X}) = Var\left(\frac{1}{3}X_1 + \frac{1}{3}X_2 + \frac{1}{3}X_3\right) = \left(\frac{1}{3}\right)^2 Var(X_1) + \left(\frac{1}{3}\right)^2 Var(X_2) + \left(\frac{1}{3}\right)^2 Var(X_3) = \frac{1}{3}\sigma^2$$

$$MSE(\hat{\theta}) = Var(\hat{\theta}) + [Bias(\hat{\theta})]^2$$

$MSE(\tilde{X}) = \frac{5}{9}\sigma^2$ and $MSE(\overline{X}) = \frac{1}{3}\sigma^2$, since $Bias(\tilde{X}) = Bias(\overline{X}) = 0$

17.4

Both $\overline{X}$ and $\tilde{X}$ are unbiased estimator where $Var(\tilde{X}) > Var\left(\overline{X}\right)$ (and $MSE(\tilde{X}) > MSE\left(\overline{X}\right)$)

Hence, $\overline{X}$ is a better estimator than $\tilde{X}$.