

EE320 (1/2017)

INTRODUCTORY MATHEMATICAL ECONOMICS

DERIVATIVES OF MORE-THAN-ONE INDEPENDENT
VARIABLE FUNCTION

Roadmap after midterm

- Derivatives of more-than-one independent variable functions
- Optimization without constraint: More-than-one independent variable cases
- Optimization under equality constraint
- Integration and applications

Topics

- Partial Differentiation
 - First-order partial derivatives
 - Second-order partial derivatives
- Differentials
- Total differentials
- Total derivatives
- Implicit function and its derivative

Introduction

- This lecture illustrates calculus tools used to deal with functions with more than one independent variables. Consider the following functions:

$$Q_d = (P_1, P_2, Y)$$

$$Q(K, L) = AK^\alpha L^{1-\alpha}$$

$$U = (c, l) \quad \text{where} \quad w(24 - l) = pc$$

- **Partial differentiation** is used to find the impact of a change in one independent variable on the dependent variable, holding everything else constant, we need to use.
- **Total differential** is used to find the total change in the dependent variable as one or more independent variables change at the same time.
- **Total derivative** is used when independent variable are related to one another.

PARTIAL DIFFERENTIATION

Partial Differentiation

- Consider $y = f(x_1, x_2, \dots, x_n)$ where $x_1, \dots, x_n$ are independent of one another.
- Suppose only x_1 changes by Δx_1 , the corresponding change in y is Δy .

- Difference quotient:

$$\frac{\Delta y}{\Delta x_1} = \frac{f(x_1 + \Delta x_1, x_2, \dots, x_n) - f(x_1, x_2, \dots, x_n)}{\Delta x_1}$$

- Partial derivative of y with respect to x :

$$f_1 \equiv \frac{\partial y}{\partial x_1} \equiv \lim_{\Delta x_1 \rightarrow 0} \frac{\Delta y}{\Delta x_1}$$

- The process of taking partial derivatives is called “partial differentiation”.

First-Order Partial Derivatives

- Technique of partial differentiation: allow one variable to vary while holding all other independent variables constant while.
- Example 1: $y = f(x_1, x_2) = 3x_1^2 + x_1x_2 + 4x_2^2$. Find $f_1(1,3)$ and $f_2(1,3)$.
- Example 2: $y = f(u,v) = (u+4)(3u+2v)$. Find $f_u(2,1)$ and $f_v(2,1)$.

Geometric Interpretation of Partial Derivatives

- Suppose $Q = Q(K, L)$
 - $\partial Q / \partial L = \text{MPP}_L$
 - $\partial Q / \partial K = \text{MPP}_K$

Gradient Vector

- The **gradient vector** of the function $f(x_1, x_2, \dots, x_n)$ is an **n -vector** of all the partial derivatives:

$$\nabla f(x_1, x_2, \dots, x_n) = (f_1, f_2, \dots, f_n)$$

where $f_i \equiv \frac{\partial y}{\partial x_i}$.

- Example: Find the gradient vector of the production function $Q = aK^bL^{1-b}$, where $a > 0$ and $0 < b < 1$.

Application 1: Partial Market Equilibrium

- Recall the one-commodity market model:

$$\text{Demand:} \quad Q = a - bP \quad (a, b > 0)$$

$$\text{Supply:} \quad Q = -c + dP \quad (c, d > 0)$$

- Solutions:

$$P^* = \frac{a + c}{b + d}$$

$$Q^* = \frac{ad - bc}{b + d}$$

- Comparative-static derivatives* are the partial derivatives of P^* and Q^* with respect to parameters (a , b , c , and d):
 - $\partial P^*/\partial a$, $\partial P^*/\partial b$, $\partial P^*/\partial c$, $\partial P^*/\partial d$
 - $\partial Q^*/\partial a$, $\partial Q^*/\partial b$, $\partial Q^*/\partial c$, $\partial Q^*/\partial d$

Application 1: Partial Market Equilibrium

- Partial derivatives of P^* with respect to parameters a , b , c , and d are:

$$\frac{\partial P^*}{\partial a} = \frac{1}{b+d}$$

$$\frac{\partial P^*}{\partial b} = \frac{-(a+c)}{(b+d)^2}$$

$$\frac{\partial P^*}{\partial c} = \frac{1}{b+d} \left(= \frac{\partial P^*}{\partial a} \right)$$

$$\frac{\partial P^*}{\partial d} = \frac{-(a+c)}{(b+d)^2} \left(= \frac{\partial P^*}{\partial b} \right)$$

Application 1: Partial Market Equilibrium

- Partial derivatives of Q^* with respect to parameters a , b , c , and d are:

$$\frac{\partial Q^*}{\partial a} = \frac{d}{b+d}$$

$$\frac{\partial Q^*}{\partial b} = \frac{-(cd + ad)}{(b+d)^2}$$

$$\frac{\partial Q^*}{\partial c} = \frac{-b}{b+d}$$

$$\frac{\partial Q^*}{\partial d} = \frac{ab - bc}{(b+d)^2}$$

Application 2: Elasticities

- **Two variables**

If $z = f(x, y)$, the **partial elasticity of z w.r.t. x and y** are:

$$\varepsilon_{zx} = \frac{\partial z / z}{\partial x / x} = \frac{\partial z}{\partial x} \left(\frac{x}{z} \right) \quad \text{and} \quad \varepsilon_{zy} = \frac{\partial z / z}{\partial y / y} = \frac{\partial z}{\partial y} \left(\frac{y}{z} \right)$$

When all variables are positive, elasticities can be expressed as logarithmic derivatives:

$$\varepsilon_{zx} = \frac{\partial \ln z}{\partial \ln x} \quad \text{and} \quad \varepsilon_{zy} = \frac{\partial \ln z}{\partial \ln y}$$

- **n variables**

If $z = f(x_1, x_2, \dots, x_n) = f(\mathbf{x})$, the **elasticity of z w.r.t. x_i when all other variables are held constant** is:

$$\varepsilon_{zi} = \frac{\partial f(\mathbf{x}) / f(\mathbf{x})}{\partial x_i / x_i} = \frac{\partial \ln z}{\partial \ln x_i}$$

Application 2: Elasticities of Demand

- Given the demand function $Q_1 = a - bP_1 + cP_2 + mY$

where Y = income, P_2 = the price of a substitute good.

- Own price elasticity of demand:

$$\varepsilon_d = \frac{\partial Q_1}{Q_1} \div \frac{\partial P_1}{P_1} = \frac{\partial Q_1}{\partial P_1} \left(\frac{P_1}{Q_1} \right)$$

- Income elasticity of demand:

$$\varepsilon_Y = \frac{\partial Q_1}{Q_1} \div \frac{\partial Y}{Y} = \frac{\partial Q_1}{\partial Y} \left(\frac{Y}{Q_1} \right)$$

- Cross-price elasticity of demand:

$$\varepsilon_c = \frac{\partial Q_1}{Q_1} \div \frac{\partial P_2}{P_2} = \frac{\partial Q_1}{\partial P_2} \left(\frac{P_2}{Q_1} \right)$$

Application 2: Elasticities of Demand

- **Example** : Given $Q_1 = 100 - P_1 + 0.75P_2 - 0.25P_3 + 0.0075Y$.

At $Y = 10,000$, $P_1 = 10$, $P_2 = 20$, $P_3 = 40$ and $Q_1 = 100$, find the different **cross-price elasticities of demand**.

$$\text{➤ } \varepsilon_{12} = \frac{\partial Q_1 / Q_1}{\partial P_2 / P_2} = 0.75 \left(\frac{20}{100} \right) = 0.15$$

$$\text{➤ } \varepsilon_{13} = \frac{\partial Q_1 / Q_1}{\partial P_3 / P_3} = -0.25 \left(\frac{40}{100} \right) = -0.1$$

Application 2: Output Elasticity

- Given a *linearly homogenous* Cobb-Douglas production function

$$Q = F(K, L) = AK^\alpha L^\beta$$

- The output elasticity of capital:

$$\varepsilon_{QK} = \frac{\partial Q / Q}{\partial K / K} = \alpha AK^{\alpha-1} L^\beta \frac{K}{Q} = \alpha$$

- The output elasticity of labor:

$$\varepsilon_{QL} = \frac{\partial Q / Q}{\partial L / L} = \beta AK^\alpha L^{\beta-1} \frac{L}{Q} = \beta$$

Application 3: Production Function

- Example: Given a production function

$$Q = 36KL - 2K^2 - 3L^2$$

- Marginal product of capital is: $MP_K = 36L - 4K$
- Marginal product of labor is: $MP_L = 36K - 6L$
- If the marginal revenue (MR) at $K = 2$ and $L = 3$ is \$5, the marginal revenue product (MRP) for the *third unit of L* is:

$$\begin{aligned}\rightarrow MRP_{L=3} &= MR \times MP_L \\ &= \$5 \times [36(2) - 6(3)] \\ &= \$5 \times 54 = 270\end{aligned}$$

Application 4:

Multipliers in Macroeconomic Models

- Consider a national-income model

$$Y = C + I_0 + G_0$$

$$C = \alpha + \beta(Y - T) \quad (\alpha > 0; 0 < \beta < 1)$$

$$T = \gamma + \delta Y \quad (\gamma > 0; 0 < \delta < 1)$$

- Equilibrium national income (in reduced form):

$$Y^* = \frac{\alpha - \beta\gamma + I_0 + G_0}{1 - \beta + \beta\delta}$$

- Comparative-static derivatives:

$$\frac{\partial Y^*}{\partial \alpha}; \frac{\partial Y^*}{\partial \beta}; \frac{\partial Y^*}{\partial I_0}; \frac{\partial Y^*}{\partial G_0}; \frac{\partial Y^*}{\partial \gamma}; \frac{\partial Y^*}{\partial \delta}$$

Application 4:

Multipliers in Macroeconomic Models

- Comparative-static derivatives with special *policy significance*:

- Government-expenditure multiplier:

$$\frac{\partial Y^*}{\partial G_0} = \frac{1}{1 - \beta + \beta\delta} > 0$$

- Nonincome-tax multiplier:

$$\frac{\partial Y^*}{\partial \gamma} = \frac{-\beta}{1 - \beta + \beta\delta} < 0$$

- Partial derivatives of Y^* w.r.t. the income tax rate (δ):

$$\frac{\partial Y^*}{\partial \delta} = \frac{-\beta(\alpha - \beta\gamma + I_0 + G_0)}{(1 - \beta + \beta\delta)^2} = \frac{-\beta Y^*}{1 - \beta + \beta\delta} < 0$$

Second-Order Partial Derivatives (1)

- Consider the function $z = f(x, y)$, which give rise to:

$$f_x \equiv \frac{\partial z}{\partial x} \quad \text{and} \quad f_y \equiv \frac{\partial z}{\partial y}$$

- Since f_x is a function of x (and y), we can determine the rate of change of f_x with respect to x , while y is fixed, by a *second-order partial derivative with respect to x* :

$$f_{xx} \equiv \frac{\partial}{\partial x}(f_x) \quad \text{or} \quad \frac{\partial^2 z}{\partial x^2} \equiv \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial x} \right)$$

- Similarly, the *second-order partial derivative with respect to y* is:

$$f_{yy} \equiv \frac{\partial}{\partial y}(f_y) \quad \text{or} \quad \frac{\partial^2 z}{\partial y^2} \equiv \frac{\partial}{\partial y} \left(\frac{\partial z}{\partial y} \right)$$

Second-Order Partial Derivatives (2)

- Also, since f_x is a function of y and f_y is a function of x , *cross (or mixed) partial derivatives* can be written as:

$$f_{xy} \equiv \frac{\partial^2 z}{\partial x \partial y} \equiv \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial y} \right) \quad \text{and} \quad f_{yx} \equiv \frac{\partial^2 z}{\partial y \partial x} \equiv \frac{\partial}{\partial y} \left(\frac{\partial z}{\partial x} \right)$$

Young's Theorem:

Let $y = f(x_1, x_2, \dots, x_n)$ is twice continuously differentiable (C^2).

Then,

$$\frac{\partial^2 y}{\partial x_i \partial x_j} = \frac{\partial^2 y}{\partial x_j \partial x_i}; i \neq j$$

Second-Order Partial Derivatives (3)

- Let $y = f(x_1, x_2)$. The second-order partial derivatives can be written in a matrix form called “**Hessian matrix**”:

$$H = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix}$$

- Example: Let $z = x^2 e^{-y}$. Find f_x, f_y , and the Hessian matrix.

Application: Production Function

- Consider an agricultural production function

$$Q = F(K, L, T) = AK^\alpha L^\beta T^\gamma \quad (A > 0; \alpha > 0, \beta > 0, \gamma > 0)$$

Where K = capital, L = labor, and T = land.

- Marginal product of capital is: $F_K = A\alpha K^{\alpha-1} L^\beta T^\gamma$
- Marginal product of labor is: $F_L = A\beta K^\alpha L^{\beta-1} T^\gamma$
- Marginal product of land is: $F_T = A\gamma K^\alpha L^\beta T^{\gamma-1}$
- Second-order partial derivatives:
- Cross partial derivatives:

DIFFERENTIALS & TOTAL DIFFERENTIALS

Differentials

- Recall the definition of derivatives:

$$\frac{dy}{dx} = f'(x) = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x}$$

$$\Rightarrow dy = f'(x)dx$$

➤ dy = differential of y

➤ dx = differential of x

- Example: Let $y = 3x^2 + 7x - 5$. Find dy .
- The process of finding the differential dy from a function $y = f(x)$ is called *differentiation*. (Note: for finding dy/dx , this is called differentiation with respect to x).

Total Differentials

- **Total differential** of a function Y is the **sum of the approximated changes from all parameters**.
- The process of finding total differentials is called “**total differentiation**.”

- Let $y = f(x_1, x_2)$

$$dy = f_1 dx_1 + f_2 dx_2, \quad \text{where } f_1 = \partial y / \partial x_1, f_2 = \partial y / \partial x_2$$

- Example: Let $z = 5x^2 + 6y + 7$. Find $dz = ?$

Example 1: Saving Function

- Consider a saving function $S = S(Y, r)$ where Y = income, r = interest rate.

- The total change in S is approximated by the total differential:

$$dS = \left(\frac{\partial S}{\partial Y} \right) \cdot dY + \left(\frac{\partial S}{\partial r} \right) \cdot dr \quad \text{or} \quad dS = S_Y dY + S_r dr$$

- For a given change in Y , the resulting change in S can be approximated by: $dS = \left(\frac{\partial S}{\partial Y} \right) \cdot dY$

where $\frac{\partial S}{\partial Y}$ = marginal propensity to save.

- For a given change in r (dr), the resulting change in S can be approximated by: $dS = \left(\frac{\partial S}{\partial r} \right) \cdot dr$

Example 2: Utility Function

- General case of n independent variables:

Example: $U = U(x_1, x_2, \dots, x_n)$

- Total differential of U is:

$$dU = \left(\frac{\partial U}{\partial x_1} \right) \cdot dx_1 + \left(\frac{\partial U}{\partial x_2} \right) \cdot dx_2 + \dots + \left(\frac{\partial U}{\partial x_n} \right) \cdot dx_n$$

or

$$dU = U_1 dx_1 + U_2 dx_2 + \dots + U_n dx_n = \sum_{i=1}^n U_i dx_i$$

- Example: Let $U = 100x^{0.5}y^{0.5}$. Find $dU = ?$

$$\rightarrow dU = (50x^{-0.5}y^{0.5})dx + (50x^{0.5}y^{-0.5})dy$$

Rules of Differentials

Rule I $dk = 0$ (k is a constant.)

Rule II $d(cu^n) = cnu^{n-1}du$

Rule III $d(u \pm v) = du \pm dv$

Rule IV $d(u \cdot v) = v \cdot du + u \cdot dv$

Rule V $d(u/v) = [v \cdot du - u \cdot dv]/v^2$

Rule VI $d(u \pm v \pm w) = du \pm dv \pm dw$

Rule VII $d(uvw) = vw \cdot du + uw \cdot dv + uv \cdot dw$

Examples: Rules of Differentials

- Example 1: $y = 3x_1^2 + x_1x_2^2$
- Example 2: $y = \frac{x_1 + x_2}{2x_1^2}$
- Example 3: $y = 3x_1(2x_2 - 1)(x_3 + 5)$

Application: Utility Function

- Consider a utility function

$$U = Ax^a y^b,$$

- Marginal utility of x: $MU_x = aAx^{a-1}y^b$
- Marginal utility of y: $MU_y = bAx^a y^{b-1}$
- Marginal rate of substitution (MRS)** as the slope of the indifference curve:

$$MRS_{xy} = \left| \frac{dy}{dx} \right| = \left| - \frac{MU_x}{MU_y} \right| = \frac{ay}{bx}$$

Application: Production Function

- Consider an agricultural production function

$$Q = F(K, L) = 60K^{0.25}L^{0.75}$$

➤ Marginal product of capital is:

➤ Marginal product of labor is:

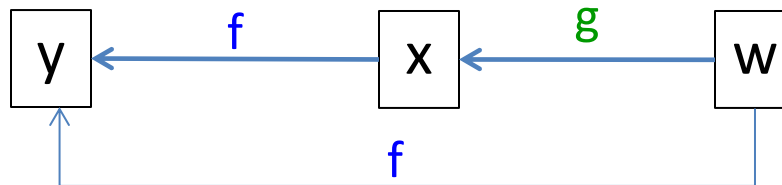
- Marginal rate of technical substitution (MRTS) as the slope of the isoquant:

$$MRTS_{KL} = \left| \frac{dK}{dL} \right| = \left| -\frac{MP_L}{MP_K} \right| = \frac{3K}{L}$$

TOTAL DERIVATIVES

Total Derivatives (1)

- Now, consider the case where independent variables are *related to one another*.
 - Example: what is the rate of change of $C(Y^*, T_0)$ w.r.t. T_0 , where Y and T_0 are related?
- Consider $y = f(x, w)$ where $x = g(w)$



➔ $y = f[g(w), w]$

➔ $dy/dw = ?$

Total Derivatives (2)

- From $y = f[g(w), w]$

1. Use chain rule:

$$\frac{dy}{dw} = f_x \frac{dx}{dw} + f_w = \frac{\partial y}{\partial x} \frac{dx}{dw} + \frac{\partial y}{\partial w}$$

2. Totally differentiate:

$$dy = f_x \cdot dx + f_w \cdot dw$$

⇒ $\boxed{\frac{dy}{dw} = f_x \frac{dx}{dw} + f_w}$

- dy/dw is called the “total derivative of y w.r.t. w ”.
- $\partial y / \partial w$ (partial derivative) is a component of the total derivative.

Examples: Total Derivatives

- Example 1: $y = f(x, w) = 3x - w^2$, where $x = g(w) = 2w^2 + w + 4$

$$\rightarrow \frac{dy}{dw} = 10w + 3$$

- Example 2: $U = U[c, g(c)]$

$$\rightarrow \frac{dU}{dc} = U_c + U_g \cdot \frac{dg}{dc}$$

A Variation on Total Derivatives

- Let $y = f(x_1, x_2, w)$ where $\begin{cases} x_1 = g(w) \\ x_2 = h(w) \end{cases}$

$$\rightarrow \frac{dy}{dw} = \frac{\partial y}{\partial x_1} \frac{dx_1}{dw} + \frac{\partial y}{\partial x_2} \frac{dx_2}{dw} + \frac{\partial y}{\partial w}$$

- Example: $Q = Q(K, L, t)$, where $K = K(t)$ and $L = L(t)$

$$\rightarrow \frac{dQ}{dt} = \frac{\partial Q}{\partial K} \frac{dK}{dt} + \frac{\partial Q}{\partial L} \frac{dL}{dt} + \frac{\partial Q}{\partial t}$$

Another Variation on Total Derivatives

- Let $y = f(x_1, x_2, u, v)$ where
$$\begin{cases} x_1 = g(u, v) \\ x_2 = h(u, v) \end{cases}$$

Application: Utility Function

- Suppose $U = U(C, n)$,

where C = consumption, n = leisure = $24-L$, and $C = f(L)$.

IMPLICIT FUNCTION

Implicit Function

- Consider $y = f(x) = 3x^4$: Explicit function
 $\Rightarrow y - 3x^4 = 0$: Implicit function
- General form: $F(y, x_1, x_2, \dots, x_n) = 0$.
- This function may define an implicit function:

$$y = f(x_1, x_2, \dots, x_n)$$

Derivatives of Implicit Functions

- Given $F(y, x_1, x_2, \dots, x_n) = 0$,

- We can write $dF = d(0) = 0$,

or $F_y dy + F_1 dx_1 + F_2 dx_2 + \dots + F_n dx_n = 0. \quad -- (*)$

- Implicit function has the total differential:

$$dy = f_1 dx_1 + f_2 dx_2 + \dots + f_n dx_n$$

- Substituting dy in $(*)$ gives:

$$(F_y f_1 + F_1) dx_1 + (F_y f_2 + F_2) dx_2 + \dots + (F_y f_n + F_n) dx_n = 0$$

$$F_y f_i + F_i = 0 \quad \text{for } i = 1, 2, \dots, n.$$



$$f_i \equiv \frac{\partial y}{\partial x_i} = -\frac{F_i}{F_y}$$

: Implicit Function Rule

Example: Derivatives of Implicit Functions

- Example 1: Find dy/dx for the implicit function $y - 3x^4 = 0$.
- Example 2: Find $\partial y/\partial x$ for any implicit function(s) that may be defined by $F(y, x, w) = y^3x^2 + w^3 + yxw - 3 = 0$.
- Example 3: Assume $F(Q, K, L) = 0$ implicitly defines the production function $Q = f(K, L)$. Express MPP_K and MPP_L in relation to the function F .

Application 1: Partial Market Equilibrium

- Suppose now Q_d is a function of P as well as Y_0 .

$$Q_d = Q_s$$

$$Q_d = D(P, Y_0) \quad (\partial D / \partial P < 0; \partial D / \partial Y_0 > 0)$$

$$Q_s = S(P) \quad (\partial S / \partial P > 0)$$

- Equilibrium condition:

$$D(P, Y_0) - S(P) = 0.$$

- Equilibrium price:

$$P^* = P^*(Y_0).$$

- Equilibrium identity:

$$D(P^*, Y_0) - S(P^*) \equiv 0, \quad \text{where } D(P^*, Y_0) - S(P^*) = F(P^*, Y_0)$$

Application 1: Partial Market Equilibrium

- Comparative-static derivatives:

$$\text{➤ } \frac{dP^*}{dY_0} = -\frac{\partial F / \partial Y_0}{\partial F / \partial P^*} = -\frac{\partial D / \partial Y_0}{\partial D / \partial P^* - dS / dP^*} > 0$$

$$\text{➤ } \frac{dQ^*}{dY_0} = \frac{dS}{dP^*} \left(\frac{dP^*}{dY_0} \right) > 0$$

Application 2: Production Function

- Example: Given the equation for a production isoquant

$$f(K,L) = 16K^{0.25}L^{0.75} = 2144,$$

Use the implicit function rule to find the slope of the isoquant ($|dK/dL|$: marginal rate of technical substitution).

$$MRTS = \left| \frac{dK}{dL} \right| = \left| - \frac{F_L}{F_K} \right| = \frac{3K}{L}$$