

# EE415: Game Theory

## Game Theory: Mixed Strategies

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## Mixed strategies and von Neumann-Morgenstern preferences

- So far what we have considered are **pure strategy** equilibria, in which players choose deterministic actions.
- Now we consider **mixed strategy** equilibria, in which players can randomize over their actions.
- This means we need to deal with preferences regarding lotteries, i.e., the **vNM preferences**—*preferences regarding lotteries over action profiles that may be represented by the expected value of a payoff function over action profiles.*

## Mixed strategies and von Neumann-Morgenstern preferences

- In pure strategy equilibria, we deal with **ordinal preferences**, which only specify the order of your preferences, not how much you prefer one item over another.
- With **vNM preferences**, the payoff numbers in a game state the intensity of your preferences, not just the order, and you can take expectations over the numbers.

## von Neumann-Morgenstern Expected Utilities

- The following tables represent the same game with ordinal preferences but different games with vNM preferences.

	S	B
S	2, 2	0, 3
B	3, 0	1, 1

	S	B
S	8, 8	0, 11
B	11, 0	1, 1

- With vNM preferences, we can derive **expected utilities** over lotteries:  $U(p_1, \dots, p_K) = \sum_{k=1}^K p_k u(a_k)$ , where  $a_k$  is the  $k$ th outcome of the lottery, and  $p_k$  is the probability that  $a_k$  will happen.
- You prefer the lottery  $(p_1, \dots, p_K)$  to the lottery  $(p'_1, \dots, p'_K)$  only if  $\sum_{k=1}^K p_k u(a_k) > \sum_{k=1}^K p'_k u(a_k)$ .

## Mixed strategy Nash equilibrium

- A **mixed strategy** of a player in a strategic game is a probability distribution over the player's actions, denoted by  $\alpha_i(a_i)$ ; e.g.,  $\alpha_i(\text{left}) = \frac{1}{3}$ ,  $\alpha_i(\text{right}) = \frac{2}{3}$ .
  - ▷ A pure strategy is a mixed strategy that assigns probability 1 to a particular action.
- The mixed strategy profile  $\alpha^*$  in a strategic game is a **mixed strategy Nash equilibrium** if

$$U_i(\alpha_i^*, \alpha_{-i}^*) \geq U_i(\alpha_i, \alpha_{-i}^*), \forall \alpha_i \text{ and } i,$$

where  $U_i(\alpha)$  is player  $i$ 's expected payoff with the mixed strategy profile  $\alpha$ .

- Using best response functions,  $\alpha^*$  is a mixed strategy NE iff  $\alpha_i^*$  is in  $B_i(\alpha_{-i}^*)$  for every player  $i$ .

## Matching Pennies reconsidered

- There is no pure strategy Nash equilibrium in Matching Pennies.

		Player 2	
		head	tail
Player 1	head	1, -1	-1, 1
	tail	-1, 1	1, -1

- But there is a mixed strategy NE for the game with the above vNM preferences:  $((\text{head}, \frac{1}{2}; \text{tail}, \frac{1}{2}), (\text{head}, \frac{1}{2}; \text{tail}, \frac{1}{2}))$ .
- **Theorem** (Nash 1950): Every finite strategic game with vNM preferences has a mixed strategy Nash equilibrium.

## Strict domination with mixed strategies

- Player  $i$ 's mixed strategy  $\alpha_i$  **strictly dominates** her action  $a'_i$  if  $U_i(\alpha_i, a_{-i}) > u_i(a'_i, a_{-i})$  for every list  $a_{-i}$  of the other players' actions.  $a'_i$  is **strictly dominated**.
- In the following game, player 1 has no action that is dominated by a pure strategy. But action T is dominated by the mixed strategy  $(M, p; B, 1 - p)$ , with  $\frac{1}{4} < p < \frac{2}{3}$ .

		Player 2	
		L	R
Player 1	T	1, $a$	1, $b$
	M	4, $c$	0, $d$
	B	0, $e$	3, $f$

- A strictly dominated action is not used in any mixed strategy Nash equilibrium.

## Weak domination with mixed strategies

- Player  $i$ 's mixed strategy  $\alpha_i$  **weakly dominates** action  $a'_i$  if  $U_i(\alpha_i, a_{-i}) \geq u_i(a'_i, a_{-i})$  for every list  $a_{-i}$  of the other players' actions, and  $U_i(\alpha_i, a_{-i}) > u_i(a'_i, a_{-i})$  for some list  $a_{-i}$  of the other players' actions.
- A weakly dominated action, however, may be used in a mixed strategy NE.
- But *every finite strategic game has a mixed strategy NE in which no player's strategy is weakly dominated.*

## Characterization of mixed strategy NE in finite games

**A characterization for finite strategic games:** a mixed strategy profile  $\alpha^*$  is a mixed strategy NE iff, for each player  $i$ ,

- ① the expected payoff, given  $\alpha_{-i}^*$ , to every action to which  $\alpha_i^*$  assigns positive probability is the same ( $\Rightarrow$  otherwise  $i$  should just play the more profitable action rather than mixing it with other actions);
  - ▷ In other words, other players' equilibrium mixed strategies keep you indifferent between a set of your actions.
- ② the expected payoff, given  $\alpha_{-i}^*$ , to every action to which  $\alpha_i^*$  assigns zero probability is lower or at most equal to the expected payoff to any action to which  $\alpha^*$  assigns positive probability ( $\Rightarrow$  otherwise  $i$  should play that action).

## Method for finding all mixed strategy NE

- ① Eliminate strictly dominated actions from the game
- ② For each player  $i$ , choose a subset  $S_i$  of her set  $A_i$  of actions
- ③ Check if there is a mixed strategy profile  $\alpha$  that (1) assigns positive probability only to actions in  $S_i$ , and (2) satisfies the two conditions in the previous characterization
- ④ Repeat the analysis for every other collection of subsets of the players' sets of actions

## Example

- Consider the variant of the Battle of Sexes game below. What are the mixed strategy NE?

		Player 2		
		B	S	X
Player 1	B	4, 2	0, 0	0, 1
	S	0, 0	2, 4	1, 3

- By inspection, we see there is no dominated strategy to be eliminated. Further, (B, B) and (S, S) are two pure strategy equilibria.

## Four possible kinds of mixed strategy equilibrium

- What about an equilibrium in which player 1 plays a pure strategy (B or S), while player 2 plays a strictly mixed strategy? Condition 1 of the characterization impossible to meet.
- Similar reasoning rules out the potential equilibrium in which player 2 plays pure strategy while player 1 randomize over her two actions.
- What about an equilibrium in which player 1 mixes over her two actions, while player 2 mixes over two of her three actions: B & S, B & X, or S & X?
- What about an equilibrium in which player 1 mixes over her two actions, and player 2 mixes over her three actions?

## Analyzing the example (1)

- Let player 1's probability of playing B be  $p$  (hence  $1 - p$  for S).
- Player 2 mixes over B and S:
  - ▷ To satisfy conditions 1 and 2 we need
$$2p = 4(1 - p) \geq p + 3(1 - p). \text{ Impossible to hold.}$$
- Player 2 mixes over B and X:
  - ▷ To satisfy the two conditions we need
$$2p = p + 3(1 - p) \geq 4(1 - p) \Rightarrow p = \frac{3}{4}.$$
  - ▷ Next we should examine player 2's randomization. Let  $q$  be her probability of choosing B (hence  $1 - q$  for X). For player 1 to be indifferent between her two actions (condition 1; condition 2 moot here),  $4q + 0 = 0 + (1 - q) \Rightarrow q = \frac{1}{5}$ .
  - ▷ Thus  $((B, \frac{3}{4}; S, \frac{1}{4}), (B, \frac{1}{5}; S, 0; X, \frac{4}{5}))$  is a mixed strategy NE.

## Analyzing the example (2)

- Player 2 mixes over S and X:
  - ▷ In this case player 1 will always choose S. So no NE in which player 1 mixes over B and S.
- Player 2 mixes over B, S, and X
  - ▷ For player 2 to be indifferent between her three actions (condition 1; condition 2 moot here), we need  $2p = 4(1 - p) = p + 3(1 - p) \Rightarrow$  impossible.
- The NE are the two pure strategy equilibria and the strictly mixed strategy NE  $((B, \frac{3}{4}; S, \frac{1}{4}), (B, \frac{1}{5}; S, 0; X, \frac{4}{5}))$ .

## A three-player example

- Player 1 chooses between rows, player 2 chooses between columns, and player 3 chooses between tables.

		2			
		A		B	
1	A	1, 1, 1	0, 0, 0	A	
	B	0, 0, 0	0, 0, 0		
		A			
		2			
		A		B	
3	A	0, 0, 0	0, 0, 0	B	
	B	0, 0, 0	4, 4, 4		
		B			

## Analyzing the three-player example

- By inspection  $(A, A, A)$  and  $(B, B, B)$  are two pure-strategy NE.
- If one of the players' strategy is pure, obviously the other two should choose the first player's action rather than mix over two or more actions.
- The only remaining case is all three mix over A and B. Let  $p$ ,  $q$ , and  $r$  respectively denote the three players' probability of choosing A. Then condition 1 of the characterization requires
  - ①  $qr = 4(1 - q)(1 - r)$ ;
  - ②  $pr = 4(1 - p)(1 - r)$ ;
  - ③  $pq = 4(1 - p)(1 - q)$ .
- Therefore  $p = q = r = \frac{2}{3}$  is a mixed strategy NE.

## Characterization of mixed strategy NE in infinite games

- Finite games must have a mixed strategy NE. Infinite games may or may not have one.
- Condition 1 of the characterization in finite games does not apply in infinite games because the probabilities are now assigned to sets of actions, not single actions.
- **A characterization for infinite strategic games:** a mixed strategy profile  $\alpha^*$  is a mixed strategy NE iff, for each player  $i$ ,
  - ▷ for no action  $a_i$  does the action profile  $(a_i, \alpha_{-i}^*)$  yield player  $i$  an expected payoff greater than her expected payoff to  $\alpha^*$ ;
  - ▷  $\alpha^*$  assigns probability zero to the set of actions  $a_i$  for which the action profile  $(a_i, \alpha_{-i}^*)$  yields player  $i$  an expected payoff less than her expected payoff from  $\alpha^*$ .