Assignment 5

1. (Optional) Let $X = \{2, 3, 4\}$ and $Y = \{1, 5\}$. Define relations
   $f : X \to Y$ by $f = \{(x, y) \in X \times Y | x > y\}$,
   $g : X \to Y$ by $g = \{(x, y) \in X \times Y | x = 2y\}$, and
   $h : X \to Y$ by $h = \{(x, y) \in X \times Y | y = x^2\}$.

   (a) List all the elements of the Cartesian product $X \times Y$.

   (b) List all the elements of the relations f, g, and h.

   (c) Determine which of the relations f, g, h is a function from X to Y. Explain your answer.

   (d) What is the inverse image of 1 under f? What is the inverse image of 5 under f?

2. Let $A = \{1, 2, 3\}$ and let $\mathcal{P}(A)$ be the set of all subsets of the set A and let X be a subset of $\mathcal{P}(A)$, which defined as
   $$X = \{x \in \mathcal{P}(A) |\ x \cap \{1\} = \emptyset\}.$$
   Define relations r and s as
   $$r = \{(x, y) \in X \times \mathcal{P}(A) \mid \text{ } x \text{ and } y \text{ have the same number of elements } \},$$
   $$s = \{(u, v) \in \mathcal{P}(A) \times X \mid (v, u) \in r\}.$$

   (a) Determine the sets $\mathcal{P}(A)$ and X.

   (b) Draw arrow diagrams of r and s.

   (c) Is r a function? If so, is it onto and/or one-to-one? Justify your answer.

   (d) Is s a function? If so, is it onto and/or one-to-one? Justify your answer.

3. Define $H : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ as follows:
   $$H(x, y) = (y/2, x - 1) \text{ for all } (x, y) \in \mathbb{R} \times \mathbb{R}.$$

   (a) Is H one-to-one? Prove or give a counterexample.

   (b) Is H onto? Prove or give a counterexample.

   (c) Is H bijective? If so, find H^{-1}, the inverse function of H.

4. (Optional) Define functions f and g as follows:
   $f = \{(1, 10), (3, 30), (5, 50)\}$ and
   $g = \{(10, k), (20, \ell), (30, m), (40, n), (50, t)\}$.

   (a) Determine the domain and range for each of functions f and g.

   (b) Find $g \circ f$ and $f \circ g$ (if possible) and the corresponding domain and range for each of them.

   (c) Find $g \circ g^{-1}$ and $g^{-1} \circ g$ (if possible) and their corresponding domains and ranges.

5. Define
   $$g(x) = \sqrt{x^2 + 1} \qquad \text{and} \qquad f(x) = \begin{cases} \frac{x}{2}, & x < 2 \\ x - 1, & x \geq 2. \end{cases}$$

(a) Find the domain and range for each of the functions g and f.

(b) Construct the composite functions $f \circ g$, and $g \circ f$ (if possible). Determine the domains for these composite functions.

(c) (Optional) Is the function f bijective? If so, find the **inverse function** of f. Justify your answer.

Additional Optional Problems

1. Let f and g be functions from $\mathbb{R}$ to $\mathbb{R}$. Find $f \circ g$, $g \circ f$, and determine whether or not $f \circ g = g \circ f$ for the given formulas for f and g. Compute $(f \circ g)(2)$ and $(g \circ f)(2)$.

   (a) $f(x) = \frac{x}{\sqrt{x^2+1}}, \quad g(x) = x^3 + 1.$

   (b) $f(x) = x^5, \quad g(x) = x^{1/5}.$

2. Let $f : \mathbb{R} - \{1\} \to \mathbb{R} - \{-2\}$ be a function defined by $f(x) = \frac{2x+1}{1-x}$.

   (a) Show that f is bijective.

   (b) Determine the inverse function f^{-1}.

   (c) Compute $f \circ f$ and $f \circ f^{-1}$.

3. Define $f : \mathbb{Z} \to \mathbb{Z}$ by the rule $f(n) = 2 - 3n$, for all integers n.

   (i) Is f one-to-one? Prove or give a counterexample.

   (ii) Is f onto? Prove or give a counterexample.

4. Define $G : \mathbb{R} \to \mathbb{R}$ by the rule $G(x) = 2 - 3x$ for all real numbers x. Is G onto? Prove or give a counterexample.

5. Define $f : \mathbb{R} - \{0\} \to \mathbb{R}$ by $f(x) = \frac{x+1}{x}$, for all real numbers $x \neq 0$. Determine whether or not f is one-to-one and justify your answer.

6. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \frac{x}{x^2+1}$, for all real numbers x. Determine whether or not f is one-to-one and justify your answer.

7. Define $G : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ as follows: $G(x, y) = (2y, -x)$ for all $(x, y) \in \mathbb{R} \times \mathbb{R}$.

   a. Is G one-to-one? Prove or give a counterexample.

   b. Is G onto? Prove or give a counterexample.

8. You work forty hours a week at a furniture store. You receive a \$220 weekly salary, plus a 3% commission on sales over \$5000. Assume that you sell enough this week to get the commission. Given the functions $f(x) = 0.03x$ and $g(x) = x - 5000$, which of $(f \circ g)(x)$ and $(g \circ f)(x)$ represents your commission?

9. You make a purchase at a local hardware store, but what you've bought is too big to take home in your car. For a small fee, you arrange to have the hardware store deliver your purchase for you. You pay for your purchase, plus the sales taxes, plus the fee. The taxes are 7.5

(i) Write a function $t(x)$ for the total, after taxes, on the purchase amount x. Write another function f(x) for the total, including the delivery fee, on the purchase amount x.

(ii) Calculate and interpret $(f \circ t)(x)$ and $(t \circ f)(x)$. Which results in a lower cost to you?

(iii) Suppose taxes, by law, are not to be charged on delivery fees. Which composite function must then be used?

10. Your computer's screen saver is an expanding circle. The circle starts as a dot in the middle of the screen and expands outward, changing colors as it grows. With a twenty-one inch screen, you have a viewing area with a 10-inch radius (measured from the center diagonally down to a corner). The circle reaches the corners in four seconds. Express the area of the circle (discounting the area cut off by the edges of the viewing area) as a function of time t in seconds.