Solution: Assignment 2 (Part 3)

1. Determine the truth value of each of these statements. Explain your answer.

   (a) $\forall n \in \mathbb{Z},\quad n^3 > n$ (b) $\exists n \in \mathbb{Z},\quad n^2 + n = 1$ (c) $\exists n \in \mathbb{R},\quad n^2 + n = 1$
   (d) $\forall n \in \mathbb{R}^+,\quad 2n > \frac{1}{n}$ (e) $\forall n \in \mathbb{R}, (n > 0) \to (3n > n)$

   Answer:

   (a) $\forall n \in \mathbb{Z},\quad n^3 > n$
   False because we can find a counterexample $n = -3$.
   $$n^3 = -27 \not> -3 = n.$$
   Note any negative integer could be used as a counterexample.

   (b) $\exists n \in \mathbb{Z},\quad n^2 + n = 1$
   False. Note that $n^2 + n = 1$ is true only when
   $$n^2 + n - 1 = 0 \qquad \text{or } n = \frac{-1 \pm \sqrt{1 - 4(1)(-1)}}{2} = \frac{-1 \pm \sqrt{5}}{2}.$$
   That is, $n^2 + n - 1 = 0$ when $n = \frac{-1+\sqrt{5}}{2}$ or $n = \frac{-1-\sqrt{5}}{2}$, which are not in the set of integers $\mathbb{Z}$. Therefore, there is no value in $\mathbb{Z}$ that makes $n^2 + n = 1$ true.

   (c) $\exists n \in \mathbb{R},\quad n^2 + n = 1$
   True. From (b), we see that $n^2 + n - 1 = 0$ when $n = \frac{-1+\sqrt{5}}{2}$ or $n = \frac{-1-\sqrt{5}}{2}$, which are in the set of real numbers $\mathbb{R}$. Hence the statement is true.

   (d) $\forall n \in \mathbb{R}^+,\quad 2n > \frac{1}{n}$
   False because we can find a counterexample $n = 1/2$: $2n = 1$ and $\frac{1}{n} = 2$ and $2n \not> \frac{1}{n}$, or
   $$2n = 2\frac{1}{2} = 1 \not> 2 = \frac{1}{1/2} = \frac{1}{n}.$$
   Note any positive rational number that is less than 1 could be used as a counterexample, e.g $n = 1/3, 2/3, 1/5$.

   (e) $\forall n \in \mathbb{R}, (n > 0) \to (3n > n)$
   True. To show that this universal statement is true we need to consider all possible value of n in $\mathbb{R}$. Let $P(n)$ be $R(n) \to S(n)$, where $R(n) = (n > 0)$ and $S(n) = (3n > n)$. When $n > 0$, then we have

   (i) For $n > 0$, $R(n)$ is true and $S(n) = (3n > n) \equiv 3 > 1$(by dividing both side by $n > 0$), which is always true, i.e. "$T \to T$"
   Hence, $P(n)$ is true.

   (ii) For $n \leq 0$, $R(n)$ is false and $S(n) = (3n > n)$ is false, i.e. "$F \to F$"
   Hence, $P(n)$ is (vacuously) true.

   Also notice that
   $$\begin{aligned} 3n &> n \\ 3n - n &> 0 \\ 2n &> 0 \\ n &> 0. \end{aligned}$$
   That is, $3n > n$ in equivalent to $n > 0$.

2. Let $\mathbb{R}$ be the domain of x. Determine the **truth set** for each of these statements.
   (a) $P(x)$: "$x < \frac{1}{x}$" (b) $P(x)$: "$2x + 1 < 0$ or $x \geq 1$"

   Answer:

   (a) $P(x)$: "$x < \frac{1}{x}$"

   $$x < \frac{1}{x} \quad \Leftrightarrow \quad x - \frac{1}{x} < 0 \quad \Leftrightarrow \quad \frac{x^2 - 1}{x} < 0$$

   which can occur when either one of the following conditions happen.

   (i) when $x^2 - 1 < 0$ and $x > 0$, i.e.
   $-1 < x < 1$ and $x > 0$, or $x \in (-1, 1) \cap (0, \infty) = (0, 1)$

   (ii) when $x^2 - 1 > 0$ and $x < 0$, i.e.
   ($x < -1$ or $x > 1$) and $x > 0$, or $x \in \{(-\infty, -1) \cup (1, \infty)\}(-1, 1) \cap (-\infty, 0) = (-\infty, -1)$.

   Since the truth set will contain all the possible values of $x \in \mathbb{R}$ that makes $P(x)$ true, we have from (i) and (ii), $P(x)$ is true when $x \in (0, 1) \cup (-\infty, -1)$ the truth set is $(-\infty, -1) \cup (0, 1)$.

   (b) $P(x)$: "$2x + 1 < 0$ or $x \geq 1$"
   Since $2x + 1 < 0 \equiv x < -\frac{1}{2}$, $x \in (-\infty, \frac{1}{2})$. That is, $P(x)$ is true either when $x \in (-\infty, \frac{1}{2})$ or $x \in [1, \infty)$. Therefore, the truth set is $(-\infty, \frac{1}{2}) \cup [1, \infty)$.

3. Let the domain for variables x and y be the set of real numbers $\mathbb{R}$. Determine the truth values of the following statements. Explain you answer.
   (a) $\exists y \forall x, \ \ xy = x$ (b) $\forall x \exists y, \ \ xy = x$ (c) $\forall y \exists x, \ \ y = x$ (d) $\exists x \forall y, \ \ y = x$

   Answer:

   (a) **True**. The statement $\exists y \forall x, \ \ xy = x$ is **true** since we can use $y = 1 \in \mathbb{R}$, then we have $x = x$ which is always true for all $x \in \mathbb{R}$

   (b) **True**. Consider $\forall x \exists y, \ \ xy = x$. To show that this statement is true, we need to consider all possible values for $x \in \mathbb{R}$, which will be dividing into two cases: $x \neq 0$ and $x = 0$.

   (i) For $x \neq 0$, "$xy = x$" gives $y = 1$ (by diving both sides by x). That is, we have that for any $x \in \mathbb{R}$ with $x \neq 0$, there is $y = 1 \in \mathbb{R}$ such that $xy = x$.

   (ii) For $x = 0$, "$xy = x$" gives $0y = 0$, which implies that y can be any real number to have this statement true (and hence there is at least one value of). That is, we have that for any $x \in \mathbb{R}$ with $x \neq 0$, there is $y = 1 \in \mathbb{R}$ such that $xy = x$.

   From cases (i) and (ii) we have that for any given $x \in \mathbb{R}$, there is $y \in \mathbb{R}$ such that $xy = x$.

   (c) $\forall y \exists x, \ \ y = x$
   True. Given any fixed value of $y \in \mathbb{R}$, we can find $x \in \mathbb{R}$ such that

   $$y = x$$

   by setting $x = y$. Therefore this is true.

   (d) $\exists x \forall y, \ \ y = x$
   False. Suppose we fix a value of $x \in \mathbb{R}$. Let $y = x + 1$. So there exists $y = x + 1 \in \mathbb{R}$ such that

   $$y \neq x \qquad (\text{ since } y = x + 1 \neq x)$$

for any given $x \in \mathbb{R}$. Therefore, we cannot find $x \in \mathbb{R}$ such that every value of $y \in \mathbb{R}$ makes $x = y$.

4. Let $Q(x, y, z)$ be the statement "$xy = z$." If the domain for variables x, y, z is the set of all integers, determine the truth values of the following statements. Explain you answer.
   (a) $Q(1, 2, 2)$ (b) $Q(2, 0, 2)$ (c) $\exists y, Q(2, y, 1)$
   (d) $\forall x \forall y \exists z, Q(x, y, z)$ (e) $\exists z \forall x \forall y, Q(x, y, z)$

   Answer:

   (a) $Q(1, 2, 2)$
   $Q(1, 2, 2) : 1 \cdot 2 = 2$ is **true**.

   (b) $Q(2, 0, 2)$
   $Q(2, 0, 2) : 2 \cdot 0 = 2$ is **false**.

   (c) $\exists y, Q(2, y, 1)$
   $\exists y \in \mathbb{Z}, 2y = 1$ is **false** because the equation $2y = 1$ is true only when $y = \frac{1}{2}$, which is **not** in the set of integers $\mathbb{Z}$.

   (d) $\forall x \forall y \exists z, Q(x, y, z)$ is **true** .
   Given any fixed $x, y \in \mathbb{Z}$, we can set $z = xy$ to make this statement true.

   (e) $\exists z \forall x \forall y, Q(x, y, z)$ is **false**.
   Given any fixed $z \in \mathbb{Z}$, it is impossible to have $xy = z$, for all $x, y \in \mathbb{Z}$.
   To show that this statement is not true, we can show that its negation :

   $$\sim (\exists z \forall x \forall y, z = xy) \equiv \forall z \exists x \exists y, z \neq xy$$

   is true. We need to consider all possible values of integers z. Here we will divide the values of $z \in \mathbb{Z}$ into 3 cases:
   (i) For $z = 1$, we can set $x = 0 \in \mathbb{Z}$ and $y = 0 \in \mathbb{Z}$, so we have $xy = 0$ and $z \neq xy$.

   (ii) For $z = 0$, we can set $x = 1$ and $y = 1$ so that $z \neq xy$.

   (iii) For any fixed value $z \in \mathbb{Z} - \{0, 1\}$, we can set
   $x = z \in \mathbb{Z}$ and $y = z \in \mathbb{Z}$, so that we have $xy = z^2$, which implies $z^2 = z$. Note that $z^2 = z$ is true only when $z = 0$ or $z = 1$, which are not included in this case. Therefore, for $z \in \mathbb{Z} - \{0, 1\}$, we can find $x \in \mathbb{Z}$ and $y \in \mathbb{Z}$ such that $z \neq xy$.

   From above, we have shown that its negation is true and therefore the statement itself is **false**.

5. Let $\mathbb{Z}^+$ be the domain of x. Let $P(x)$ and $Q(x)$ be the predicates "x is not divisible by 3," and "x is divisible by 12," respectively. Determine whether the following statements are true or false. Give a counterexample for each false statement.
   (a) $Q(x) \Rightarrow P(x)$ (b) $P(x) \Rightarrow \sim Q(x)$

   Answer:

   The truth set of of $P(x)$ is

   $$T_P := \{x \in \mathbb{Z}^+ | x \text{ is not divisible by } 3\} = \mathbb{Z}^+ - \{3, 6, 9, 12, 15, 18, 21, 24, 27, 30, 33, 36, ...\}.$$

The truth set of of $Q(x)$ is

$$T_Q := \{x \in \mathbb{Z}^+ | x \text{ is divisible by } 12\} = \{12, 24, 36, 48, 60, 72, ...\}.$$

(a) **False**. $T_Q \nsubseteq T_P$.

A counterexample is when $x = 36 \in T_Q$ but $x = 36 \notin T_P$.
I,e. $Q(36)$ is true , but $P(36)$ is not true. Hence, $Q(x) \Rightarrow P(x)$ is false.

(b) **True**. The truth set of $\sim Q(x)$ is $\mathbb{Z}^+ - \{12, 24, 36, ...\}$, which contains Tp. I.e., $T_P \subseteq T_{\sim Q}$. Hence, $P(x) \Rightarrow \sim Q(x)$ is true.

6. Write a negation for each statement without using *the negation symbol* "∼."

   (a) $\exists z \forall x \forall y,\ xy = z$

   (b) $\forall x \forall y,\ (x < 0) \wedge (y \geq 0) \rightarrow (xy \leq 0)$

   Answer: (a) The negation of $\exists z \forall x \forall y,\ xy = z$ can be found step-by-step as follows.

$$\begin{aligned} \sim (\exists z \forall x \forall y,\ xy = z) &\equiv \forall z \sim (\forall x \forall y,\ xy = z) \\ &\equiv \forall z \exists x \sim (\forall y,\ xy = z) \\ &\equiv \forall z \exists x \exists y \sim (\ xy = z) \\ &\equiv \forall z \exists x \exists y, \quad xy \neq z \end{aligned}$$

(b) The negation of $\forall x \forall y,\ (x < 0) \wedge (y \geq 0) \rightarrow (xy \leq 0)$ can be found step-by-step as follows. Consider $p \wedge q \rightarrow r$, using the order of operation we have $(p \wedge q) \rightarrow r$ and its negation is

$$\sim ((p \wedge q) \rightarrow r) \equiv (p \wedge q) \wedge \sim r.$$

From above, that the negation of $(x < 0) \wedge (y \geq 0) \rightarrow (xy \leq 0)$ is

$$(x < 0) \wedge (y \geq 0) \wedge \sim (xy \leq 0) \equiv (x < 0) \wedge (y \geq 0) \wedge (xy > 0).$$

The negation of $\forall x \forall y,\ (x < 0) \wedge (y \geq 0) \rightarrow (xy \leq 0)$ is

$$\begin{aligned} \sim (\forall x \forall y,\ (x < 0) \wedge (y \geq 0) \rightarrow (xy \leq 0)) &\equiv \exists x \sim (\forall x,\ (x < 0) \wedge (y \geq 0) \rightarrow (xy \leq 0)) \\ &\equiv \exists x \exists y, \sim (\ (x < 0) \wedge (y \geq 0) \rightarrow (xy \leq 0)) \\ &\equiv \exists x \exists y, (x < 0) \wedge (y \geq 0) \wedge (xy > 0). \end{aligned}$$

7. Show that each of the following arguments is valid by **universal modus ponens** , **universal modus tollens** and/or **universal transitivity**, or show that it is invalid from the **converse error** or the **inverse error**. In addition, use also the **diagram** to confirm that each argument is valid or invalid.

(a)

"Anyone who has a school email account has a school ID number."
"Kevin has a school ID number."
∴ "Kevin has a school email account. "

(b)

"Anyone who has a school email account has a school ID number."
"All students have school email accounts."
"Kim does not have a school ID number."
∴ "Kim is not a student."

Answer:

(a) "Anyone who has a school email account has a school ID number."
"Kevin has a school ID number."
∴ "Kevin has a school email account. "

Let $P(x)$ be "x has a school email account,"
$Q(x)$ be "x has a school ID number."
Then we can transform the given argument in the quantified form of **converse error** as follows.

$$\begin{aligned} &\forall x, P(x) \to Q(x) \\ &Q(a) \text{ for a particular } a = \text{Kevin} \\ &\therefore P(a) \end{aligned}$$

That is, this argument is **invalid** by the converse error.

To use the diagram,
let A be the set of people who have school email accounts, and
let B be the set of people who have school ID numbers.
Then
- the first premise tells us that "$A \subseteq B$."
- the second premise tells us that " Kevin is in the set B,"
- the conclusion tells us that "Kevin is in the set A."

From the diagram, since we only know that "Kevin" is an element in B , it is possible that "Kevin" is either an element in A or not an element in A. That is, the conclusion may not happen because it is possible that "Kevin" does **not** have an email account. Therefore the statement is **invalid**.

(b) "Anyone who has a school email account has a school ID number."
"All students have school email accounts."
"Kim does not have a school ID number."
∴ "Kim is not a student."

Let $P(x)$ be "x has a school email account,"
$Q(x)$ be "x has a school ID number,"
$R(x)$ be "x is a student."
Then we can transform the given argument in the quantified form as follows.

$$\begin{aligned} &\forall x, P(x) \to Q(x) \\ &\forall x, R(x) \to P(x) \\ &\sim Q(a) \text{ for a particular } a = \text{Kim} \\ &\therefore \sim R(a) \end{aligned}$$

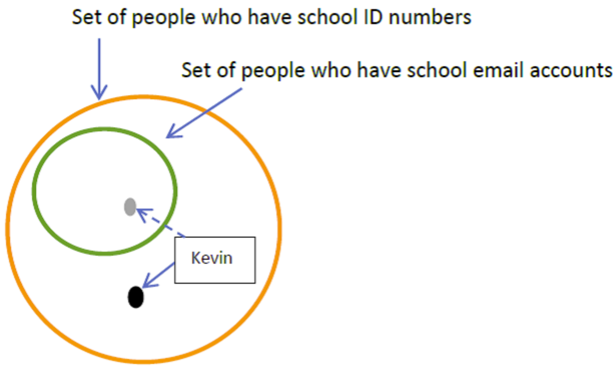

Figure 1: Problem 7 (a)

and using the universal transitivity rule we have

$$\begin{array}{l} \forall x, R(x) \rightarrow P(x) \\ \forall x, P(x) \rightarrow Q(x) \\ \therefore R(x) \rightarrow Q(x). \end{array}$$

Using the result from the transitivity rule above, $\forall x, R(x) \rightarrow Q(x)$, the original argument becomes

$$\begin{array}{l} \forall x, R(x) \rightarrow Q(x) \\ \sim Q(a) \text{ for a particular } a= \text{Kim} \\ \therefore\sim R(a) \end{array}$$

which is valid by universal modus tollens. That is, the original argument is **vilid** by **the universal transitivity and universal modus tollens**.
To use the diagram,
let A be the set of people who have school email accounts,
let B be the set of people who have school ID numbers, and
let C be the set of students.
Then
the first premise tells us that "$A \subseteq B$."
the second premise tells us that "$C \subseteq A$."
the third premise tells us that "Kim is **not** in the set B,"
the conclusion tells us that "Kim is **not** in the set C."

That is, the second and the third premise gives $C \subseteq A \subseteq B$, which implies that $C \subseteq B$. From the diagram, since "Kim " is not an element in B, it is **impossible** that "Kim " is an element in C, which is inside B. I.e., the conclusion that "Kim is not an element in C" is **always** true. Hence, the given argument is **valid**.

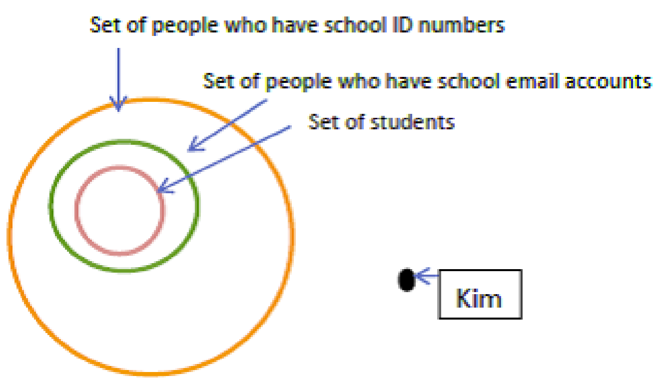

Figure 2: Problem 7 (b)