

## Solution: Assignment 6

1. Consider the relation  $f$  defined from  $X$  to  $Y$ ,  $X, Y \subseteq \mathbb{R}$ ,

$$f = \left\{ (x, y) \in X \times Y \mid yx^3 = x^2 - 1 \right\}.$$

- (a) Find the domain of  $f$ .
- (b) Find  $x$ -intercepts and  $y$ -intercepts (if any).
- (c) Determine the symmetry of  $f$ .
- (d) Find the horizontal and vertical asymptotes for  $f$  (if any).
- (e) Find the critical number of  $f$ . Determine the intervals on which  $f$  is increasing and decreasing. Determine the relative extrema (maximum and minimum) of  $f$ .
- (f) Determine the intervals on which  $f$  is concave up and concave down. Find the points of inflection (if any).
- (g) Sketch the curve of  $f$ .

### Answer:

- (a) Notice that we can rearrange as  $y = \frac{x^2-1}{x^3}$ . So, the domain of  $f$  (which consists of possible values of  $x$ ) is  $\mathbb{R} - \{0\}$ .
- (b)  $x$ -intercepts: setting  $y = 0$  gives  $0 = \frac{x^2-1}{x^3}$ , which occurs if and only if

$$x^2 - 1 = 0 \quad \Leftrightarrow \quad (x - 1)(x + 1) = 0 \quad \Leftrightarrow \quad x = -1, 1.$$

So  $x$ -intercepts are  $(-1, 0), (1, 0)$ .

To find  $y$ -intercepts, set  $x = 0$ . However,  $0$  is not in the domain. So There is **no  $y$ -intercepts**.

- (c) Determine the symmetry of  $f$ .  
Notice that  $f$  is a function and can be written as  $f(x) = \frac{x^2-1}{x^3}$ . Hence, it is not symmetric about the  $x$ -axis. We will look at the symmetry by using  $f(-x)$ .

$$f(-x) = \frac{(-x)^2 - 1}{(-x)^3} = -\frac{x^2 - 1}{x^3}.$$

Since  $f(-x) = -f(x)$ ,  $f$  is an odd function and it is symmetric about the origin.

Note that  $f(-x) \neq f(x)$ , it is not symmetric about the  $y$ -axis.

- (d) Find the horizontal and vertical asymptotes for  $f$  (if any).

### Horizontal asymptotes:

$$\lim_{x \rightarrow -\infty} f(x) = \lim_{x \rightarrow -\infty} \frac{x^2 - 1}{x^3} = 0$$

$$\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} \frac{x^2 - 1}{x^3} = 0$$

So, the horizontal asymptote is  $y = 0$ .

**Vertical asymptotes:**

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} \frac{x^2 - 1}{x^3} = -\infty$$

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} \frac{x^2 - 1}{x^3} = +\infty.$$

So, the vertical asymptote is  $x = 0$ .

(e) Find the critical number of  $f$ .

$$f'(x) = \frac{x^3(2x) - 3x^2(x^2 - 1)}{x^6} = \frac{2x^4 - 3x^4 + 3x^2}{x^6} = \frac{-x^2 + 3}{x^4}$$

The critical number  $x$  occurs when  $f'(x) = 0$  or  $f'(x)$  does not exist.

$$f'(x) = 0 \iff -x^2 + 3 = 0 \iff x = -\sqrt{3}, \sqrt{3}.$$

Note that  $f'(x)$  does not exist when  $x = 0$ , which is not in the domain, so 0 is not a critical number. Hence, the critical numbers are  $x = -\sqrt{3}, \sqrt{3}$ .

(f) Determine the intervals on which  $f$  is increasing and decreasing. Determine the relative extrema (maximum and minimum) of  $f$ .

From

$$f'(x) = \frac{-x^2 + 3}{x^4} = -\frac{(x + \sqrt{3})(x - \sqrt{3})}{x^4}$$

- Decreasing:  $f'(x) < 0$  when  $-(x + \sqrt{3})(x - \sqrt{3}) < 0$  or  $(x + \sqrt{3})(x - \sqrt{3}) > 0$
- Increasing:  $f'(x) > 0$  when  $-(x + \sqrt{3})(x - \sqrt{3}) > 0$  or  $(x + \sqrt{3})(x - \sqrt{3}) < 0$

	$x < -\sqrt{3}$	$-\sqrt{3} < x < \sqrt{3}$	$\sqrt{3} < x$
Sign of $f'(x) = \frac{-(x+\sqrt{3})(x-\sqrt{3})}{x^4}$	-	+	-
$-(x + \sqrt{3})(x - \sqrt{3})$	$-(-)(-)$	$-(+)(-)$	$-(+)(+)$

I.e.,  $f'(x) < 0$  when  $x < -\sqrt{3}$  and  $x > \sqrt{3}$ . Also,  $f'(x) < 0$  for  $x \in (-\sqrt{3}, \sqrt{3})$  or  $x \in (-\sqrt{3}, 0) \cup (0, \sqrt{3})$  (note:  $x = 0$  is not in the domain)

- Decreasing interval:  $(-\infty, -\sqrt{3})$  and  $(\sqrt{3}, \infty)$
- Increasing interval:  $(-\sqrt{3}, 0) \cup (0, \sqrt{3})$ .

Therefore, from the *First Derivative Test*,

- relative maximum occurs at  $x = \sqrt{3}$  (“+” changes to “-” at  $x = \sqrt{3}$ )

and  $f(\sqrt{3}) = \frac{2}{3\sqrt{3}}$  is the relative maximum

- relative minimum occurs at  $x = -\sqrt{3}$  (“-” changes to “+” at  $x = -\sqrt{3}$ ) and

$f(-\sqrt{3}) = -\frac{2}{3\sqrt{3}}$  is the relative minimum.

(g) Determine the intervals on which  $f$  is concave up and concave down. Find the points of inflection (if any).

From  $f'(x) = \frac{-x^2+3}{x^4}$ ,

$$f''(x) = \frac{x^4(-2x) - (3 - x^2)(4x^3)}{x^8} = \frac{-2x^5 - 12x^3 + 4x^5}{x^8} = \frac{2(x^5 - 6x^3)}{x^8} = \frac{2(x^2 - 6)}{x^5}.$$

So  $f''(x) = 0$  when  $\frac{x^2-6}{x^5} = \frac{(x-\sqrt{6})(x+\sqrt{6})}{x^5} = 0$  or  $x = \pm\sqrt{6}$  and  $f''(x)$  is undefined when  $x = 0$ . To solve the inequalities, we will also consider the denominator (because  $x^5$  changes sign for  $x < 0$  and  $x > 0$ ).

Interval of $x$	$(-\infty, -\sqrt{6})$	$(-\sqrt{6}, 0)$	$(0, \sqrt{6})$	$(\sqrt{6}, \infty)$
Sign of $f''(x) = \frac{2(x^2-6)}{x^5}$	-	+	-	+
$\frac{(x-\sqrt{6})(x+\sqrt{6})}{x^5}$	$\frac{(-)(-)}{(-)}$	$\frac{(-)(+)}{(-)}$	$\frac{(-)(+)}{(+)}$	$\frac{(+)(+)}{(+)}$

- Concave up:  $f'' > 0$  when  $x \in (-\sqrt{6}, 0) \cup (\sqrt{6}, \infty)$ .
- Concave down:  $f'' < 0$  when  $x \in (-\infty, -\sqrt{6}) \cup (0, \sqrt{6})$ .

The points of inflection are  $(\sqrt{6}, f(\sqrt{6})) = (\sqrt{6}, \frac{5}{6\sqrt{6}})$  and  $(-\sqrt{6}, f(-\sqrt{6})) = (-\sqrt{6}, -\frac{5}{6\sqrt{6}})$  because the curve changes the concavity at these points (note that 0 is not an inflection point because it is not in the domain).

(h) Sketch the curve of  $f$ .

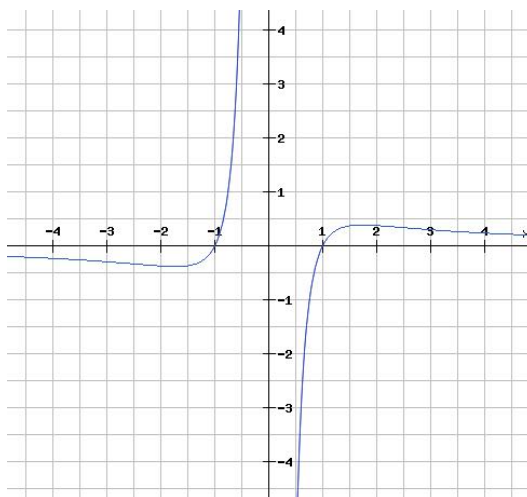


Figure 1:  $f = \{(x, y) | yx^3 = x^2 - 1\}$

### Optional Problems

1. Find the critical numbers of the given functions on  $(-\infty, \infty)$ .

(a)  $f(x) = x^3 - 3x^2 + 3x - 1$

(b)  $f(x) = \frac{x^2}{x^2+2}$

(c)  $f(x) = e^{-x} + 2x$

(d)  $f(x) = -x + \sin(x)$

(e)  $f(x) = x^2 - 8 \ln(x)$

(a)  $f(x) = x^3 - 3x^2 + 3x - 1$

Answer:  $f'(x) = 3x^2 - 6x + 3$

$$f'(x) = 0 \Leftrightarrow 3(x^2 - 2x + 1) = 3(x - 1)^2 = 0 \Leftrightarrow (x - 1) = 0 \Leftrightarrow x = 1$$

Therefore, the only critical number of  $f(x)$  is  $x = 1$ .

(b)  $f(x) = \frac{x^2}{x^2+2}$

Answer:

$$f'(x) = \frac{(x^2+2)(2x) - x^2(2x)}{(x^2+2)^2} = \frac{2x^3+4x-x^3}{(x^2+2)^2} = \frac{x^3+4x}{(x^2+2)^2}$$

$$f'(x) = 0 \text{ when } x^3+4x = 0 \implies x(x^2+4) \implies x = 0.$$

Therefore, the only critical number of  $f(x)$  is  $x = 0$ .

(c)  $f(x) = e^{-x} + 2x$

Answer:

$$f'(x) = e^{-x} + 2x = -e^{-x} + 2$$

$$f'(x) = 0 \text{ when } -e^{-x} + 2 = 0 \implies e^{-x} = 2 \implies e^x = 1/2 \implies x = \ln(1/2) \text{ or } x = -\ln(2).$$

Therefore, the only critical number of  $f(x)$  is  $x = -\ln(2)$ .

(d)  $f(x) = -x + \sin(x)$

Answer:

$$f'(x) = -1 + \cos(x)$$

$$f'(x) = 0 \text{ when } -1 + \cos(x) = 0 \implies \cos(x) = 1 \implies x = 2\pi n \text{ for } n = 0, \pm 1, \pm 2, \dots$$

Therefore, the critical numbers of  $f(x)$  are  $x = 2\pi n$ , where  $n = 0, \pm 1, \pm 2, \dots$ 

(e)  $f(x) = x^2 - 8 \ln(x)$

Answer:

$$f'(x) = 2x - \frac{8}{x}$$

$$f'(x) = 0 \text{ when } 2x - \frac{8}{x} = 0 \implies x = \frac{4}{x} \implies x^2 = 4 \implies x = \pm 2.$$

Therefore, the critical numbers of  $f(x)$  are  $x = -2$  and  $x = 2$ .

2. Determine the intervals on which the given function  $f$  is increasing and the interval on which  $f$  is decreasing.

(a)  $f(x) = x^2 + 6x - 1$

(b)  $f(x) = x^4 - 4x^3 + 9$

(c)  $f(x) = x^2 e^{-x}$

Answer: The function  $f(x)$  is increasing when  $f'(x) > 0$  and  $f(x)$  is decreasing when  $f'(x) < 0$ .

(a)  $f(x) = x^2 + 6x - 1 \implies f'(x) = 2x + 6$

$$f'(x) = 2x + 6 > 0 \implies x > -3$$

so  $f(x)$  is increasing on the interval  $[-3, \infty)$ , and

$$f'(x) = 2x + 6 < 0 \implies x < -3$$

so  $f(x)$  is decreasing on the interval  $(-\infty, -3]$ .

(b)  $f(x) = x^4 - 4x^3 + 9 \implies f'(x) = 4x^3 - 12x^2$

$$f'(x) = 4x^3 - 12x^2 > 0 \implies 4x^2(x-3) > 0 \implies x > 3$$

so  $f(x)$  is increasing on the interval  $[3, \infty)$ , and

$$f'(x) = 4x^3 - 12x^2 < 0 \implies 4x^2(x-3) < 0 \implies x < 3$$

so  $f(x)$  is decreasing on the interval  $(-\infty, 3]$ .

- (c)  $f(x) = x^2e^{-x} \implies f'(x) = -x^2e^{-x} + 2xe^{-x} \implies f'(x) = xe^{-x}(2 - x)$ ,  
 since  $e^{-x} > 0$ ,  $f'(x) = 0 \implies x = 0$  or  $x = 2$ :

	$x < 0$	$0 < x < 2$	$2 < x$
Sign of $f'(x) = xe^{-x}(2 - x)$	-	+	-
$(x)(e^{-x})(2 - x)$	(-)(+)(+)	(+)(+)(+)	(+)(+)(-)

so  $f(x)$  is increasing ( $f'(x) = +$ ) on the interval  $[0, 2]$ ,  
 and  $f(x)$  is decreasing ( $f'(x) = -$ ) on the interval  $(\infty, 0] \cup [2, \infty)$ .

3. Use the *First Derivative Test* to find the relative extrema of the given function.

- (a)  $f(x) = x^3 - 3x$       (b)  $f(x) = x^3 + x - 3$       (c)  $\frac{x^2+3}{x+1}$       (d)  $f(x) = x^2 - 2|x|$

Answer:

- (a)  $f(x) = x^3 - 3x \implies f'(x) = 3x^2 - 3 = 3(x - 1)(x + 1)$ ,  
 $f'(x) = 0 \implies x = \pm 1$  are critical numbers:

	$x < -1$	$-1 < x < 1$	$1 < x$
Sign of $f'(x) = 3(x - 1)(x + 1)$	+	-	+
$(x - 1)(x + 1)$	(-)(-)	(+)(-)	(+)(+)

Therefore, from the *First Derivative Test*,

- relative maximum occurs at  $x = -1$  (“+” changes to “-” at  $x = -1$ )

and  $f(-1) = (-1)^3 - 3(-1) = 2$  is the relative maximum

- relative minimum occurs at  $x = 1$  (“-” changes to “+” at  $x = 1$ ) and

$f(1) = 1^3 - 3(1) = -2$  is the relative minimum.

- (b)  $f(x) = x^3 + x - 3 \implies f'(x) = 3x^2 + 1 \neq 0, \forall x \in (-\infty, \infty)$ .

$f'(x)$  is well-defined for any  $x$  and  $f'(x) \neq 0$  imply that there is *no critical number* for  $f(x)$ . Since a relative extremum can only occur at a critical number and there is no critical number, therefore, we conclude that **there is no relative extrema** for this function.

- (c)  $f(x) = \frac{x^2+3}{x+1}$  Note that the domain of this function is all real number except for  $x = -1$ :  
 $D = \mathbb{R}/\{-1\}$

$$\implies f'(x) = \frac{(x+1)(2x) - (x^2+3)}{(x+1)^2} = \frac{x^2+2x-3}{(x+1)^2} = \frac{(x+3)(x-1)}{(x+1)^2},$$

$f'(x) = 0 \implies (x + 3)(x - 1) = 0 \implies x = -3, 1$  are the critical numbers.

	$x < -3$	$-3 < x < 1(x \neq -1)$	$1 < x$
Sign of $f'(x)$	+	-	+
$(x + 3)(x - 1)$	(-)(-)	(+)(-)	(+)(+)

Therefore, from the *First Derivative Test*,

- relative maximum occurs at  $x = -3$  (“+” changes to “-” at  $x = -3$ )

and  $f(-3) = \frac{(-3)^2+3}{-3+1} = -6$  is the relative maximum

- relative minimum occurs at  $x = 1$  (“-” changes to “+” at  $x = 1$ ) and

$f(1) = \frac{1^2+3}{1+1} = 2$  is the relative minimum.

Notice that it is possible to have a “relative” minimum *larger* than a “relative” maximum.

(d) Note that we can write  $f(x) = x^2 - 2|x|$  as:

- For  $x < 0$ ,  $f(x) = x^2 + 2x$
- For  $x \geq 0$ ,  $f(x) = x^2 - 2x$ .

(i) The critical numbers:

- For  $x < 0$ ,  $f'(x) = 2x + 2$
- For  $x > 0$ ,  $f'(x) = 2x - 2$ ,

and  $f'(x) = 0$  when  $2x + 2 \Rightarrow x = -1$  or  $2x - 2 = 0 \Rightarrow x = 1$ . Note that  $f'(x)$  does not exist when  $x = 0$ . Hence, the critical numbers are  $x = -1, 1, 0$ .

(ii) Relative extrema: Relative extrema can occur only at the critical points:  $x = -1, 1, 0$ . So we consider the subintervals:

	$x < -1$	$-1 < x < 0$	$0 < x < 1$	$1 < x$
	$f'(x) = 2x + 2$	$f'(x) = 2x + 2$	$f'(x) = 2x - 2$	$f'(x) = 2x - 2$
Sign of $f'(x)$	-	+	-	+

Therefore, from the *First Derivative Test*,

- relative maximum occurs at  $x = 0$  (“+” changes to “-” at  $x = 0$ )

and  $f(0) = 0$  is the relative maximum

- relative minima occurs at  $x = -1, 1$  (“-” changes to “+” at  $x = -1, 1$ ) and

$f(-1) = 1^2 - 2|-1| = -1$  and  $f(1) = 1^2 - 2|1| = -1$  are the relative minima.

That is,  $-1$  is the relative minimum, which occurs at  $x = \pm 1$ .

4. For each given function,

- (i) use the Second Derivative Test, when applicable, to find the relative extrema;
  - (ii) find intercepts and points of inflection, when possible;
  - (iii) find the intervals on which it is increasing and the interval on which it is decreasing;
  - (iv) find the intervals on which it is concave up and the intervals on which it is concave down;
- and (v) sketch the graph.

(a)  $f(x) = x^3 + 3x^2 + 3x + 1$

(b)  $f(x) = 6x^5 - 10x^3$

(c)  $f(x) = \frac{x}{x^2+2}$

(d)  $f(x) = 2x - x \ln(x)$

Answer:

Note that there are 2 steps

(1) Find critical numbers  $c_j$

(2) Check sign of  $f''(c_j)$ :

- If  $f''(c_j) < 0$ ,  $f(c_j)$  is a relative maximum.

- If  $f''(c_j) > 0$ ,  $f(c_j)$  is a relative minimum.

(If  $f''(c_j) = 0$ , no conclusion – use the First Derivative Test.)

(a)  $f(x) = x^3 + 3x^2 + 3x + 1$

$$f'(x) = 3x^2 + 6x + 3, \quad f''(x) = 6x + 6.$$

(i) Critical numbers:

$$f'(x) = 3x^2 + 6x + 3 = 0 \Rightarrow 3(x+1)^2 = 0 \Rightarrow x = -1 \text{ is the only critical number.}$$

From the Second Derivative Test,

$$f''(-1) = 6(-1) + 6 = 0$$

so we cannot conclude anything. The First derivative will be used instead,

	$x \in (-\infty, -1)$	$x \in (-1, \infty)$
Sign of $f'(x) = 3(x+1)^2$	+	+

Therefore, from the table above, the *First Derivative Test* implies that there is **no relative extremum** since the sign of  $f'(x)$  is always positive.

(ii) Find intercepts and points of inflection:

- The y-intercept ( $x=0$ ) is at  $y = f(0) = 0^3 + 3(0)^2 + 3(0) + 1 = 1 \Rightarrow y = 1$ .

- The x-intercept ( $y=0$ ) is at  $0 = f(x) = (x+1)^3 \Rightarrow x = -1$ .

- To find the inflection points, we first solve  $f''(x) = 0$  and see the sign change of  $f''(x)$ :

$$f''(x) = 6x + 6 = 0 \implies x = -1.$$

	$x \in (-\infty, -1)$	$x \in (-1, \infty)$
Sign of $f''(x) = 6(x+1)$	-	+

Since the sign of  $f''(x)$  changes at the  $x = -1$ , it gives an inflection point. When  $x = -1$ ,  $y = f(-1) = (-1)^3 + 3(-1)^2 + 3(-1) + 1 = 0$ . Hence, the inflection point is  $(-1, 0)$ .

(iii) Since  $f'(x) = 3(x+1)^2 > 0$  for all  $x \neq -1$ , and  $f'(x) = 0$  at  $x = -1$ ,  $f(x)$  is increasing on  $(-\infty, \infty)$ .

(iv) Concavity:

$$f''(x) = 6x + 6 > 0 \Rightarrow x > -1 \text{ and } f''(x) = 6x + 6 < 0 \Rightarrow x < -1.$$

That is,  $f(x)$  is concave up on  $(-1, \infty)$  and  $f(x)$  is concave down on  $(-\infty, -1)$ .

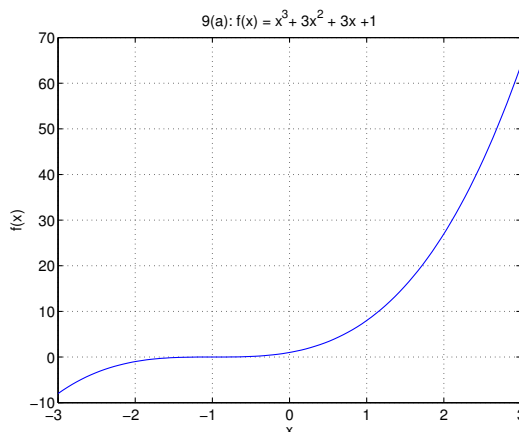


Figure 2: (a):  $f(x) = x^3 + 3x^2 + 3x + 1$

(b)  $f(x) = 6x^5 - 10x^3$

(i) Critical numbers:

$$f'(x) = 30x^4 - 30x^2 = 30x^2(x^2 - 1) = 30x^2(x - 1)(x + 1) = 0 \Rightarrow$$

$x = 0, -1, 1$  are the only critical numbers. From

$$f''(x) = 120x^3 - 60x = 60x(2x^2 - 1)$$

and from the Second Derivative Test,

$$f''(0) = 0 \Rightarrow \text{no conclusion}$$

$$f''(-1) = -60 < 0 \Rightarrow, f(-1) = -6 + 10 = 4 \text{ is a relative maximum}$$

$$f''(1) = 60 > 0 \Rightarrow, f(1) = 6 - 10 = -4 \text{ is a relative minimum.}$$

The First Derivative Test will be used to identify the critical number  $x = 0$  as shown in the table. Since the sign of  $f'(x)$  does not change at  $x = 0$ ,  $x = 0$  does not give an extremum.

	$x \in (-\infty, -1)$	$x \in (-1, 0)$	$x \in (0, 1)$	$x \in (1, \infty)$
Sign of $f'(x) = 30x^2(x - 1)(x + 1)$	+	-	-	+
$x^2(x - 1)(x + 1)$	(+)(-)(-)	(+)(-)(+)	(+)(-)(+)	(+)(+)(+)

(ii) Find intercepts and points of inflection:

- The y-intercept ( $x=0$ ) is at  $y = f(0) = 6(0)^5 - 10(0)^3 = 0 \Rightarrow y = 0$ .

- The x-intercept ( $y=0$ ) occurs when  $0 = f(x) = x^3(6x^2 - 10)$

$\Rightarrow$  at  $x = 0, x = -\sqrt{5/3}, x = \sqrt{5/3}$ .

- To find the inflection points, we first solve  $f''(x) = 0$  and see the sign change of  $f''(x)$ :

$$f''(x) = 60x(2x^2 - 1) = 0 \Rightarrow x = 0, -\sqrt{1/2}, \sqrt{1/2}.$$

	$x \in (-\infty, -\sqrt{1/2})$	$x \in (-\sqrt{1/2}, 0)$	$x \in (0, \sqrt{1/2})$	$x \in (\sqrt{1/2}, \infty)$
Sign of $f''(x) = 60x(2x^2 - 1)$	-	+	-	+
$(x)(x - \sqrt{1/2})(x + \sqrt{1/2})$	(-)(-)(+)	(-)(-)(+)	(+)(-)(+)	(+)(+)(+)

Since the sign of  $f''(x)$  changes at the  $x = 0, -\sqrt{1/2}, \sqrt{1/2}$ , they give inflection points. Since  $f(0) = 0$ ,  $f(-\sqrt{1/2}) = 7\sqrt{2}/4$ ,  $f(\sqrt{1/2}) = -7\sqrt{2}/4$ , the inflection points are  $(0, 0)$ ,  $(-\sqrt{1/2}, 7\sqrt{2}/4)$ ,  $(\sqrt{1/2}, -7\sqrt{2}/4)$ .

(iii) From the table in (i), we see that  $f(x)$  is increasing ( $f'(x) > 0$ ) on  $(-\infty, -1) \cup (1, \infty)$  and  $f(x)$  is decreasing ( $f'(x) < 0$ ) on  $(-1, 1)$ .

(iv) From the table in (ii), we see that  $f(x)$  is concave up ( $f''(x) > 0$ ) on  $(-\sqrt{1/2}, 0) \cup (\sqrt{1/2}, \infty)$  and  $f(x)$  is concave down ( $f''(x) < 0$ ) on  $(-\infty, -\sqrt{1/2}) \cup (0, \sqrt{1/2})$ .

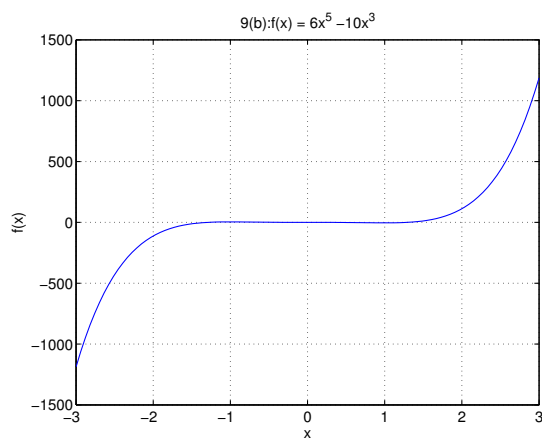


Figure 3: (b):  $f(x) = 6x^5 - 10x^3$

(c)  $f(x) = \frac{x}{x^2+2}$

(i) Critical numbers:

$$f'(x) = \frac{(x^2 + 2) - 2x^2}{(x^2 + 2)^2} = \frac{2 - x^2}{(x^2 + 2)^2} = 0 \Rightarrow 2 - x^2 = 0$$

$x = -\sqrt{2}, \sqrt{2}$  are the only critical numbers. From

$$f''(x) = \frac{2x(x^2 + 2)(x^2 - 6)}{(x^2 + 2)^4} = \frac{2x(x^2 - 6)}{(x^2 + 2)^3}$$

and from the Second Derivative Test,

$$f''(-\sqrt{2}) = \frac{\sqrt{2}}{8} > 0 \implies, f(-\sqrt{2}) = \frac{-\sqrt{2}}{(-\sqrt{2})^2+2} = -\sqrt{2}/4 \text{ is a relative minimum}$$

$$f''(\sqrt{2}) = -\frac{\sqrt{2}}{8} < 0 \implies, f(\sqrt{2}) = \frac{\sqrt{2}}{(\sqrt{2})^2+2} = \sqrt{2}/4 \text{ is a relative maximum.}$$

(ii) Find intercepts and points of inflection:

- The y-intercept(  $x=0$ ) is at  $y = f(0) = 0 \Rightarrow y = 0$ .
- The x-intercept(  $y=0$ ) occurs when  $0 = f(x) \Rightarrow$  at  $x = 0$ .
- To find the inflection points, we first solve  $f''(x) = 0$  and see the sign change of  $f''(x)$ :

$$f''(x) = \frac{2x(x^2 - 6)}{(x^2 + 2)^3} = 0 \implies x = 0, -\sqrt{6}, \sqrt{6}.$$

Since the denominator  $(x^2 + 2)^3 > 0$ , the sign of  $f''(x)$  depends on the numerator  $2x(x^2 - 6) = 2x(x - \sqrt{6})(x + \sqrt{6})$  as shown in the table below.

	$x \in (-\infty, -\sqrt{6})$	$x \in (-\sqrt{6}, 0)$	$x \in (0, \sqrt{6})$	$x \in (\sqrt{6}, \infty)$
Sign of $f''(x) = \frac{2x(x^2-6)}{(x^2+2)^3}$	-	+	-	+
$(2x)(x - \sqrt{6})(x + \sqrt{6})$	$(-)(-)(+)$	$(-)(-)(+)$	$(+)(-)(+)$	$(+)(+)(+)$

Since the sign of  $f''(x)$  changes at the  $x = 0, -\sqrt{6}, \sqrt{6}$ , they give inflection points. Since  $f(0) = 0, f(-\sqrt{6}) = \frac{-\sqrt{6}}{(-\sqrt{6})^2+2} = -\sqrt{6}/8, f(\sqrt{6}) = \frac{\sqrt{6}}{(\sqrt{6})^2+2} = \sqrt{6}/8$ , the inflection points are  $(0, 0), (-\sqrt{6}, -\sqrt{6}/8), (\sqrt{6}, \sqrt{6}/8)$ .

(iii) From

$$f'(x) = \frac{2 - x^2}{(x^2 + 2)^2} = \frac{(\sqrt{2} - x)(\sqrt{2} + x)}{(x^2 + 2)^2},$$

the denominator is always positive, so the sign of  $f'(x)$  depends on the numerator:  $(\sqrt{2} - x)(\sqrt{2} + x)$ .

	$x \in (-\infty, -\sqrt{2})$	$x \in (-\sqrt{2}, \sqrt{2})$	$x \in (\sqrt{2}, \infty)$
Sign of $f'(x) = \frac{(\sqrt{2}-x)(\sqrt{2}+x)}{(x^2+2)^2}$	-	+	-
$(\sqrt{2} - x)(\sqrt{2} + x)$	$(+)(-)$	$(+)(+)$	$(-)(+)$

Hence,  $f(x)$  is increasing ( $f'(x) > 0$ ) on  $(-\sqrt{2}, \sqrt{2})$  and  $f(x)$  is decreasing ( $f'(x) < 0$ ) on  $(-\infty, -\sqrt{2}) \cup (\sqrt{2}, \infty)$ .

(iv) From the table in (ii), we see that  $f(x)$  is concave up ( $f''(x) > 0$ ) on  $(-\sqrt{6}, 0) \cup (\sqrt{6}, \infty)$  and  $f(x)$  is concave down ( $f''(x) < 0$ ) on  $(-\infty, -\sqrt{6}) \cup (0, \sqrt{6})$ .

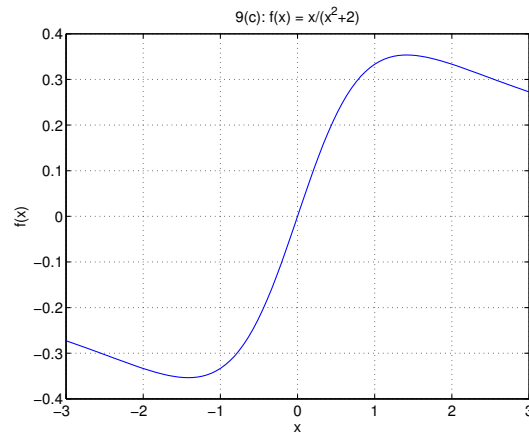


Figure 4: (c):  $f(x) = \frac{x}{x^2+2}$

(d)  $f(x) = 2x - x \ln(x)$

First note that the domain of  $f$  is

$$\mathcal{D} = (0, \infty),$$

since  $\ln(x)$  is only well-defined for  $x > 0$ .

(i) Critical numbers:  $f'(x) = 2 - [x^{\frac{1}{x}} + \ln(x)] = 1 - \ln(x)$ .

$$f'(x) = 1 - \ln(x) = 0 \Rightarrow \ln(x) = 1 \Rightarrow x = e^1.$$

$x = e$  is the only critical number. From

$$f''(x) = -\frac{1}{x}$$

and from the Second Derivative Test,

$$f''(e) = -\frac{1}{e} < 0 \implies, f(e) = 2e - e \ln(e) = 2e - e = e \text{ is a relative maximum.}$$

(ii) Find intercepts and points of inflection:

- There is no y-intercept, since  $x = 0$  is not in the domain of the function  $f(x) = 2x - x \ln(x)$  ( $\ln(x)$  is not well-defined at  $x = 0$ ).

- The x-intercept ( $y=0$ ) occurs when  $0 = f(x) \Rightarrow 2x - x \ln(x) = x(2 - \ln(x)) = 0 \Rightarrow x = 0$  or  $\ln(x) = 2 \Rightarrow$  at  $x = 0, x = e^2$ .

- To find the inflection points, we first solve  $f''(x) = 0$  and see the sign change of  $f''(x)$ : notice that

$$f''(x) = -\frac{1}{x} \neq 0 \implies \text{There is no inflection point.}$$

(iii) From  $f'(x) = 1 - \ln(x)$ ,

- $f'(x) > 0$  when  $1 - \ln(x) > 0 \Leftrightarrow \ln(x) < 1 \Leftrightarrow x < e \Rightarrow f(x)$  is **increasing** on  $(-\infty, e]$ .
- $f'(x) < 0$  when  $1 - \ln(x) < 0 \Leftrightarrow \ln(x) > 1 \Leftrightarrow x > e \Rightarrow f(x)$  is **decreasing** on  $[e, \infty)$ .

(iv) From  $f''(x) = -\frac{1}{x}$ ,

$f(x)$  is concave down ( $f''(x) < 0 \Leftrightarrow x > 0$ ) on  $(0, \infty)$ . Note that  $f(x)$  is not concave up, since  $f''(x) > 0 \Leftrightarrow x < 0$ , but  $x < 0$  is not in the domain of  $f(x)$ :  $\mathcal{D} = (0, \infty)$ .

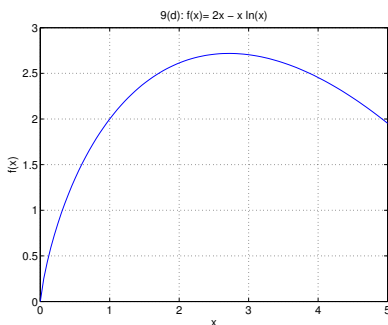


Figure 5: (d):  $f(x) = 2x - x \ln(x)$

5. Let  $X, Y_1, Y_2 \subseteq \mathbb{R}$ . Consider the relations  $f_1, f_2$  defined by

$$f_1 = \left\{ (x, y) \in X \times Y_1 \mid y = \sqrt{\frac{x+1}{2x-3}} \right\}, \quad f_2 = \left\{ (x, y) \in X \times Y_2 \mid (x, -y) \in f_1 \right\}.$$

Sketch the curve of the relation  $f = f_1 \cup f_2$ .

**Answer:**

(Details to be added...symmetry/critical numbers/increasing& decreasing/ concavity... )

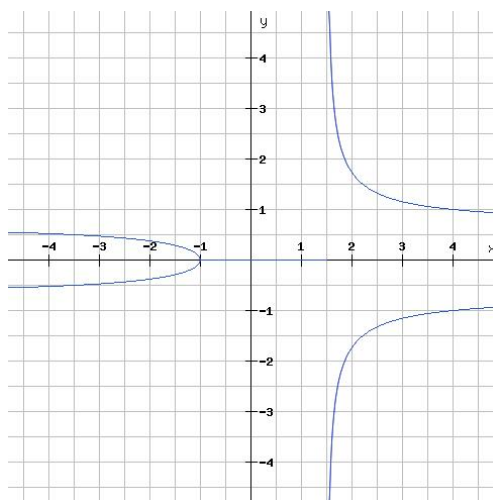


Figure 6:  $f = f_1 \cup f_2 = \{(x, y) \mid y^2 = \frac{x+1}{2x-3}\}$

6. Find the vertical and horizontal asymptotes (if any) of  $f(x) = \frac{x^2-1}{2x}$  and show that  $y = \frac{x}{2}$  is a slant asymptote of  $f$ .

**Answer:**(Details to be added...)

- Vertical asymptote:  $x = 0$
- Horizontal asymptote: None
- Slant asymptote  $y = \frac{x}{2}$ . We have to show that  $\lim_{x \rightarrow \infty} [f(x) - \frac{x}{2}] = 0$ .

$$\lim_{x \rightarrow \infty} [f(x) - \frac{x}{2}] = \lim_{x \rightarrow \infty} \left( \frac{x^2-1}{2x} - \frac{x}{2} \right) = \lim_{x \rightarrow \infty} \left( \frac{x^2-1-x^2}{2x} \right) = \lim_{x \rightarrow \infty} \frac{-1}{2x} = 0$$