EE320 (2/2012)

INTRODUCTORY MATHEMATICAL ECONOMICS

OPTIMIZATION WITHOUT CONSTRAINTS:

MORE-THAN-ONE-INDEPENDENT VARIABLE CASES

(Additional materials)

Topics

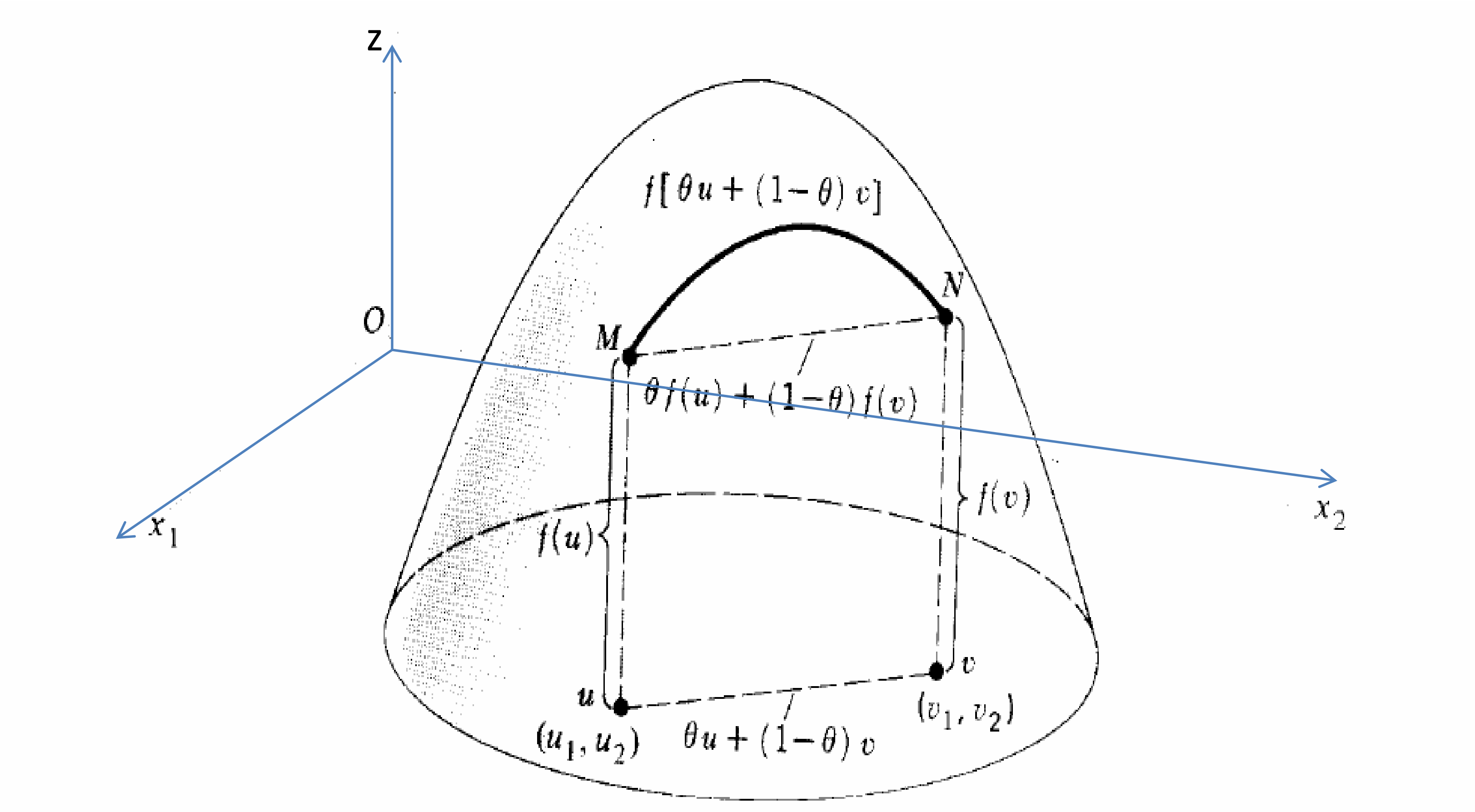
- Second-order conditions in relations to concavity and convexity
- Comparative-static aspects of optimization

Checking Concavity and Convexity

- Consider a two-variable function $z = f(x_1, x_2)$.
- **Definition**:
  - The function is *concave* (*convex*) if and only if, for any pair of distinct points M and N on its graph – surface – line segment MN lies either *on* or *below* (*above*) the surface.
  - The function is *strictly concave* (*strictly convex*) if and only if line segment MN lies *entirely below* (*above*) the surface, except at M and N.

θ

Strictly Concave Function



A function *f* is strictly concave (convex) *iff*, for any pair of distinct points *u* and *v* in the domain of f, and for 0 < θ <1,

$$\theta f(u) + (1-\theta)f(v) < (>) f[\theta(u) + (1-\theta)(v)]$$

Theorems on Concavity and Convexity

- **Theorem 1** (Linear Function)

  If *f(x)* is a linear function, then it is a concave function as well as a convex function, but not strictly so.

- **Theorem 2** (Negative of a Function)

  If *f(x)* is a concave function, then – *f(x)* is a convex function, and vice versa. Similarly, if *f(x)* is a strictly concave function, then – *f(x)* is a strictly convex function

- **Theorem 3** (Sum of Functions)

  If *f(x)* and *g(x)* are both concave (convex) functions, then *f(x) + g(x)* is also a concave (convex) function. If *f(x)* and *g(x)* are both strictly concave (strictly convex) functions, then *f(x) + g(x)* is a strictly concave (strictly convex) function.

Differentiable Functions

- Consider a one-variable function $z = f(x)$,

  A differentiable function $f(x)$ is *concave* (*convex*) iff, for any given point u and any other point v in the domain,

  $$f(v) \leq (\geq)\ f(u) + f'(u)(v-u)$$

  [Figure 11.7]

- Consider a function $f(x) = f(x_1, \ldots, x_n)$,

  A differentiable function $f(x_1, \ldots, x_n)$ is *concave* (*convex*) iff, for any given point $u = (u_1, \ldots, u_n)$and any other point $v = (v_1, \ldots, v_n)$ in the domain,

  $$f(v) \leq (\geq)\ f(u) + \sum_j f_j(u)(v_j - u_j), \quad where\ f_j(u) = \partial f / \partial x_j$$

Checking Concavity and Convexity by the Derivative Conditions

- Consider a function $z = f(x_1, ..., x_n)$, which is twice continuously differentiable. For such a function, second-order partial derivatives exist, and thus d^2z is defined.
- Concavity and convexity can then be checked by the sign of d^2z:

A twice continuously differentiable function $z = f(x_1, ..., x_n)$ is *concave (convex)* if, and only if, d^2z is everywhere *negative (positive) semidefinite*.

The said function is *strictly concave (strictly convex)* if (but *not* only if) d^2z is everywhere *negative (positive) definite*.

Examples

Check the following functions for concavity and convexity by the derivative conditions.

- **Example 1**: $z = -x^4$

- **Example 2**: $z = x_1^2 + x_2^2$

Comparative-Static Aspects of Optimization

Example: Consider a two-product firm in perfect competition.
The firm's revenue is $R = P_{10}Q_1 + P_{20}Q_2$ (P_{10} and P_{20} are exogenous)
And the cost function is $C = 2Q_1^2 + Q_1Q_2 + 2Q_2^2$.

→ Reduced-form solutions are the optimal output levels expressed in terms of exogenous variables (P_{10} and P_{20}):

$Q_1^* =$ and $Q_2^* =$

→ Comparative-statics of the model:

$\frac{\partial Q_1^*}{\partial P_{10}} =$; $\frac{\partial Q_1^*}{\partial P_{20}} =$; $\frac{\partial Q_2^*}{\partial P_{10}} =$; $\frac{\partial Q_2^*}{\partial P_{20}} =$

Comparative-Static Aspects of Optimization

- General-Function Models

Consider the input-decision problem. Given that $R = P \cdot Q(K, L)$ and $C = wL + rK$, where P, w, and r are exogenous, find $\partial L^*/\partial P$ *and* $\partial K^*/\partial P$.