Chapter 2 Differentiation

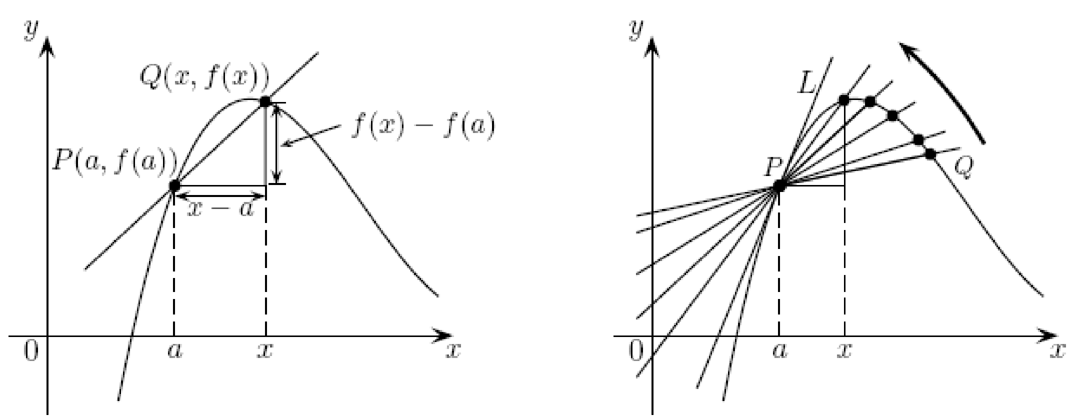
Section 2.1 Tangent Lines and Rates of Change

Tangent Lines

If a curve *C* has equation $y = f(x)$ and we want to find the tangent line to *C* at the point $P(a, f(a))$, then we consider a nearby point $Q(x, f(x))$, where $x \neq a$, and compute the slope of the secant line *PQ*:

$$m_{PQ} = \frac{f(x) - f(a)}{x - a}$$

Then we let *Q* approach *P* along the curve *C* by letting *x* approach a. If m_{PQ} approaches a number *m*, then we define the tangent *t* to be the line through *P* with slope *m*. (See figure below)



Definition 1: The tangent line to the curve $y = f(x)$ at the point $P(a, f(a))$ is the line through *P* with slope

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

provided that this limit exists.

Example 1: Find an equation of the tangent line to the curve $y = x^2 - x$ at the point (2,2).

There is another expression for the slope of a tangent line that is sometimes easier to use. If $h = x - a$, then $x = a + h$ and so the slope of the secant line PQ is

$$m_{PQ} = \frac{f(a+h) - f(a)}{h}$$

Notice that as x approaches a, h approaches 0 (because $h = x - a$) and so the expression for the slope of the tangent line in Definition 1 becomes

$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Example 2: Find an equation of the tangent line to the curve $y = 1/x$ at $x = 1$.

Rates of Change

Slope as a Derivative: The slope of the tangent line to the curve $y = f(x)$ at the point $(a, f(a))$ is $m = f'(a)$.

Instantaneous Rate of Change as a Derivative: The rate of change of $f(x)$ with respect to x when $x = a$ is given by $f'(a)$.

Marginal as a Derivative: In economics, the derivatives are often described by the adjective "marginal". For example, the derivative of a cost function is called the marginal cost function.

Section 2.2 The Derivative

We have seen that the same type of limit arises in finding the slope of a tangent line:

$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

In fact, limits of the form

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

arise whenever we calculate a rate of change in any of the sciences or engineering, such as a rate of reaction in chemistry or a marginal cost in economics. Since this type of limit occurs so widely, it is given a special name and notation.

Definition 1: The derivative of a function *f* at a number a, denoted by

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

if this limit exists. If the limit exists, we say that x is differentiable at $x = a$.

Here we change our point of view and let the number a vary, we obtain

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

<u>Example 1</u>: Find the derivative of $f(x) = \sqrt{x^2 + 1}$.

Other Notations

If we use the traditional notation $y = f(x)$ to indicate that the independent variable is *x* and the dependent variable is *y*, then some common alternative notations for the derivative are as follows:

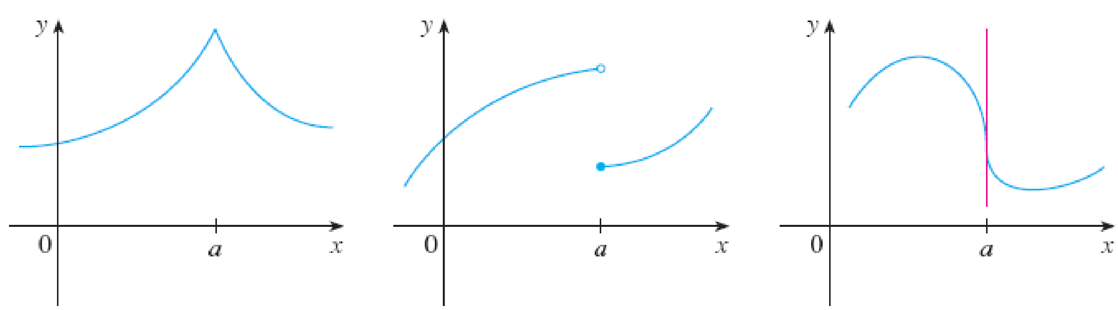
$$f'(x) = y' = \frac{dy}{dx} = \frac{d}{dx}f(x) = Df(x) = D_x f(x)$$

The symbols *D* and *d/dx* are called differentiation operators because they indicate the operation of differentiation, which is the process of calculating a derivative.

Definition: A function *f* is differentiable at *a* if $f'(a)$ exists.

How can a function fail to be differentiable?

The following figures show the cases when *f* is not differentiable at *a*.



(a) A corner (b) A discontinuity (c) A vertical tangent

Theorem: If *f* is differentiable at *a*, then *f* is continuous at *a*.

Example 2: Given $f(x) = 1/(x-1)$. Is *f* differentiable at 0? At 1?

Section 2.3 Techniques of Differentiation

If it were always necessary to compute derivatives directly from the definition, as we did in the previous section, such computations would be tedious and the evaluation of some limits would require ingenuity. Fortunately, several rules have been developed for finding derivatives without having to use definition directly.

Derivative of a Constant Function: For any constant *c*

$$\frac{d}{dx}(c) = 0$$

Power Function:

$$\frac{d}{dx}(x) = 1$$

The Power Rule: If *n* is any real number, then

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

The Constant Multiple Rule: If *c* is a constant and *f* is a differentiable function, then

$$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}[f(x)]$$

The Sum Rule: If *f* and *g* are both differentiable, then

$$\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$$

The Difference Rule: If *f* and *g* are both differentiable, then

$$\frac{d}{dx}[f(x) - g(x)] = \frac{d}{dx}f(x) - \frac{d}{dx}g(x)$$

The Product Rule: If *f* and *g* are both differentiable, then

$$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)]$$

The Quotient Rule: If *f* and *g* are both differentiable, then

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}[f(x)] - f(x)\frac{d}{dx}[g(x)]}{[g(x)]^2}$$

Example 1:

(a)If $f(x)=x^5$. Find $f'(x)$.

(b)If $y=m^7$. Find $\frac{dy}{dm}$.

(c)If $y=x^{100}$. Find y'.

(d)Find $\frac{d}{dx}\left(2x^6-x^4+3x^3-5\right)$

(e) Find y' where $y=\left(2t^2\right)\left(t^4-10\right)$.

(f)Find y' where $y=\left(x^{-3}+1\right)\left(x^2-x\right)$.

(g)Let $y=\frac{4x^5-2x^3+1}{x^2+1}$. Find y'.

(h)Let $y=\frac{\sqrt{x}}{4x^2+1}$. Find y'

(i)Find the points on the curve $y=x^4-6x^2+4$ where the tangent line is horizontal.

Section 2.4 Derivatives of Trigonometric Functions

Table of Derivatives of Trigonometric Functions

$\frac{d}{dx}(\sin x) = \cos x$	$\frac{d}{dx}\cos x = -\sin x$
$\frac{d}{dx}(\tan x) = \sec^2 x$	$\frac{d}{dx}\cot x = -\csc^2 x$
$\frac{d}{dx}(\sec x) = \sec x \tan x$	$\frac{d}{dx}\csc x = -\csc x \cot x$

<u>Example 1</u>: Find the derivative of the function.

(1) $f(x) = \sin x + 2\cos x$

(2) $f(x) = x^2 + \tan x$

(3) $y = \sin x \cos x$

Section 2.5 The Chain Rule

The Chain Rule

If *f* and *g* are both differentiable and $F = f \circ g$ is the composite function defined by $F(x) = f(g(x))$, then *F* is differentiable and F' is given by the product

$$F'(x) = f'(g(x))g'(x)$$

i.e. If $y = f(u)$ and $u = g(x)$ are both differentiable functions, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

The Power Rule Combined with the Chain Rule:

If *n* is any real number and $u = g(x)$ is differentiable, then

$$\frac{d}{dx}\left(u^n\right) = nu^{n-1}\frac{du}{dx}$$

Alternatively,

$$\frac{d}{dx}(g(x))^n = n(g(x))^{n-1} \cdot g'(x)$$

Example 1: Find the derivative of the function.

(1) $f(x) = \sqrt{x^2 + 1}$

(2) $y = \left(x^2 + 1\right)\sqrt[3]{x^2 + 2}$

(3) $y = \dfrac{r}{\sqrt{r^2 + 1}}$

(4) $y = \sin\left(\sin\left(\sin(x^2)\right)\right)$

(5) $y = \sin^2\left(\sin\left(\sqrt{x}\right)\right)$

Section 2.6 Implicit Differentiation

We have seen the functions that can be describe by expressing one variable explicitly in terms of another variable. For example,

$$y=\sqrt{x^2+1} \quad OR \quad y=x^2\sin x \quad OR \quad y=f(x)$$

Some functions, however, are defined implicitly by a relation between x and y such as

$$x^2+y^2=1 \quad OR \quad x^3+y^3=6xy \quad OR \quad x^2y+xy^2=3x$$

To differentiate this type of function, we can use method of implicit differentiation by differentiate both sides of the equation with respect to x and then solving the resulting equation for y'.

i.e.

Step 1) differentiate both sides w.r.t. x

Step 2) solve for y' or $\frac{dy}{dx}$.

Example 1: If $x^2+y^2=2$. Find dy/dx

Example 2: Find an equation of the tangent to the circle $x^2 + y^2 = 2$ at the point (1,-1).

Example 3: Find y' if $\sqrt{x} + \sqrt{y} = 4$.

Example 4: Find y' if $x^2 + y^3 = 2xy$.

<u>Example 5</u>: Find y' if $x^2y + xy^2 = 3x$.

<u>Example 6</u>: If $1 + f(x) + x^2\left(f(x)\right)^3 = 11$ and $f(1) = 2$, find $f'(1)$.

Section 2.7 Derivatives of Logarithmic and Exponential Functions

2.7.1 Derivatives of Exponential Functions

Theorem: For any constant $a > 0$

$$\frac{d}{dx}(a^x) = a^x lna$$

Proof: We use the fact that $e^{lna} = a$:

$$\frac{d}{dx}(a^x) = \frac{d}{dx}(e^{lna})^x = \frac{d}{dx}e^{(lna)x} = e^{(lna)x}\frac{d}{dx}(lna)x = e^{(lna)x}(lna) = a^x lna$$

Definition: e is the number such that

$$\lim_{h\to 0}\frac{e^h - 1}{h} = 1$$

Since $lne = 1$ and if we put $a = e$, then we have

$$\frac{d}{dx}(e^x) = e^x lne = e^x$$

Theorem: Derivative of the Natural Exponential Function

$$\frac{d}{dx}e^x = e^x \qquad and \qquad \frac{d}{dx}(e^{-x}) = -e^{-x}$$

Example 1: Differentiate the function

$$f(x) = 2^{\sqrt{x}} + \frac{e^{-4x}}{x+1}$$

2.7.2 Derivative of Logarithmic Functions

Recall:

$$log_a x = \frac{lnx}{lna}$$

Since lna is a constant, we can differentiate as follows:

$$\frac{d}{dx}(log_a x) = \frac{d}{dx}\frac{lnx}{lna} = \frac{1}{lna}\frac{d}{dx}(lnx) = \frac{1}{xlna}$$

Hence, for any constant $a > 0$

$$\frac{d}{dx}(log_a x) = \frac{1}{xlna}$$

Since $lne = 1$ and $log_e x = lnx$, and if we put $a = e$, then we have

$$\frac{d}{dx}(lnx) = \frac{d}{dx}(log_e x) = \frac{1}{xlne} = \frac{1}{x}$$

Theorem:

$$\frac{d}{dx}(lnx) = \frac{1}{x}$$

In general, if we combine the formula above with the Chain Rule, we get

$$\frac{d}{dx}(lnu) = \frac{1}{u}\frac{du}{dx} \qquad or \qquad \frac{d}{dx}[lng(x)] = \frac{g'(x)}{g(x)}$$

Example 2: Differentiate the function $y = 10^{x^2} + \log_{10}\left(2 + x^4\right)$

Example 3: Differentiate the function $y = ln\dfrac{x+1}{\sqrt{x-2}}$.

Logarithmic Differentiation

The calculation of derivatives of complicated functions involving products, quotients, or powers can often be simplified by taking logarithms. The method used in the following example is called *logarithmic differentiation*.

Steps in Logarithmic Differentiation

1. Take logarithms of both sides of an equation $y = f(x)$ and use properties of logarithms to simplify.
2. Differentiate implicitly with respect to x.
3. Solve the resulting equation for y' or $\frac{dy}{dx}$.

Example 4: Find the derivative of the function $y = x^{\sqrt{x}}$.

Example 5: Find the derivative of the function $y = (sinx)^x$.

Section 2.8 Higher Derivatives

If we take the derivative of $y = f(x)$, we get another function $f\,'(x)$ or dy/dx.

We can also differentiate that function, the result is called the *second derivative*, denoted as

$$y'' \quad OR \quad f''(x) \quad OR \quad \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) \quad OR \quad D^2 f(x)$$

Hence, the third derivative $f'''(x)$ is the derivative of the second derivative:

$$f''' = (f'')'$$

So $f'''(x)$ can be interpreted as the slope of the curve $y = f''(x)$ or as the rate of change of $f''(x)$.

Notation:

$$y''' = f'''(x) = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3} = D^3 f(x)$$

In general, we write

$$f^{(n)}(x) = \frac{d^n}{dx^n}[f(x)]$$

as the nth derivative of $f(x)$.

<u>Example 10</u>: Find the second derivative of the function.

(1) $f(x) = x^3 + 2x^2 - 1$

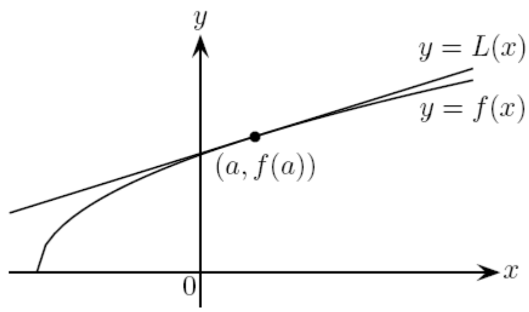
(2) $y = \dfrac{x^2+1}{\sqrt{x}}$

Section 2.9 Linear Approximations and Differential

2.9.1 Linear Approximation

We have seen that a curve lies very close to its tangent near the point of tangency. This observation is the basis for a method of finding approximate values of functions.

The idea is that it might be easy to calculate a value $f(a)$ of a function, but difficult (or even impossible) to compute nearby values of f. So we settle for the easily computed values of the linear function L whose graph is the tangent line to f at $(a, f(a))$.



In other words, we use the tangent line at $(a, f(a))$ as an approximation to the curve

$y = f(x)$ when x is near a. An equation of this tangent line is

$$y = f(a) + f'(a)(x - a)$$

and the approximation

$$f(x) = f(a) + f'(a)(x - a)$$

is called linear approximation or tangent line approximation of f at a. The linear function whose graph is this tangent line, that is,

$$L(x) = f(a) + f'(a)(x - a)$$

is called the linearization of f at a.

Example 11: Find the linearization of the function $f(x) = \sqrt{x+3}$ at a = 1 and use it to approximate the number $\sqrt{3.98}$ and $\sqrt{4.05}$.

2.11.2 Differentials

The ideas behind linear approximations are sometimes formulated in the terminology and notation of differentials. If $y = f(x)$, where f is a differentiable function, then the differential dx is an independent variable; that is, dx can be given the value of any real number. The differential dy is then defined in terms of dx by equation

$$dy = f'(x)dx$$

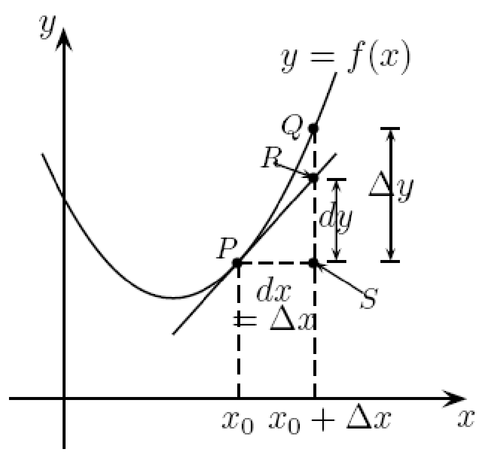
so dy is a dependent variable; it depends on the values of x and dx.

Example 12: Find the differential of the function.

$(a) y = x^3 - 2x^2$

$(b) y = \sqrt{3x^2 + x + 1}$

The geometric meaning of differentials is show in Figure below.



Let $P(x, f(x))$ and $Q(x + \Delta x, f(x + \Delta x)$ be points on the graph of f and let $dx = \Delta x$. The corresponding change in y is

$$\Delta y = f(x + \Delta x) - f(x)$$

The slope of the tangent line PR is derivative $f'(x)$. Thus the directed distance from S to R $f'(x)dx = dy$. Therefore, dy represents the amount that tangent line rises or falls (the change in the linearization), whereas Δy represents the amount that the curve $y = f(x)$ rises or falls when x changes by an amount dx.

<u>Example 13</u>: Compare the values of Δy and dy if $y = f(x) = x^3 + x^2 - 2x + 1$ and x changes (a) from 2 to 2.05 and (b) from 2 to 2.01.

Notice that the approximation $\Delta y \approx dy$ becomes better as Δx becomes smaller in Example 13. Notice also that dy was easier to compute than Δy. For more complicated functions it may be impossible to compute Δy exactly. In such cases the approximation by differentials is especially useful.

In the notation of differentials, the linear approximation can be written as

$$f(a + dx) \approx f(a) + dy$$

Example 14: Use differentials to estimate $\sqrt{4.3}$ and $\sqrt{8.6}$.

Example 15: Use differential to estimate $(1.97)^6$