Answer KeysChp4Optimization without constraint

1. a) $y= - x^2 +4x+91$

$y'= \frac{dy}{dx} = -2x+4 = 0$ then, $x = 2$ (Extremum)

$y'' = \frac{d(-2x+4)}{dx} = -2 <0$ (Maximum)

Then, $x =2$ gives Y maximum

b) $y= 2x^3+3x^2 – 12x+4$

$y' = \frac{dy}{dx} = 6x^2+6x – 12 = 0$ then, $6(x+2)(x-1) = 0$

$x= -2, 1$ (Extremum)

$y''= \frac{d(6x^2+6x-12)}{dx} = 12x+6$

At $x=-2$, $y''= 12(-2)+6= -18 <0$ (Maximum)

At $x=1$, $y''= 12(1)+6=18>0$ (Minimum)

Then, $x=-2$ gives Y maximum and $x=1$ gives Y minimum

c) $y= -3x^2+4x-2$

$y'=\frac{dy}{dx} = -6x+4 =0$ then, $x= \frac{2}{3}$ (Extremum)

$y''= \frac{d(-6x+4)}{dx} = -6$

At $x= \frac{2}{3}$, $y''= -6<0$ (Maximum)

Then, $x=\frac{2}{3}$ gives Y maximum

d) $y =(2x-7)^3$

$y'=3(2x-7)^2(2)=6(2x-7)^2=0$ then, $x=\frac{7}{2}$ (Extremum)

$y''= \frac{d6(2x-7)^2}{dx} = 12(2x-7)(2)=48x – 168$

At $x=\frac{7}{2}$, $y''= 48(\frac{7}{2})-168=0$ (Inflection point)

Then, $x=\frac{7}{2}$ gives Y at inflection point

e) $Y = \frac{1}{9}x^3 - \frac{1}{6}x^2 - \frac{2}{3}x + 1$

$y' = \frac{1}{3}x^2 - \frac{1}{3}x - \frac{2}{3} = 0$ then, $(x-2)(x+1)=0$, $x=2, -1$ (Extremum)

$y'' = \frac{d(\frac{1}{3}x^2 - \frac{1}{3}x - \frac{2}{3})}{dx} = \frac{2}{3}x - \frac{1}{3}$

At $x=2$, $y''=1 > 0$ (Minimum)

At $x=-1$, $y''=-1<0$ (Maximum)

Then, x=2 gives Y minimum and x=-1 gives Y maximum

2. a) $MU = 5Q - Q^2$

   b) $MU = 5Q - Q^2 = 0, Q(5 - Q) = 0$, then $Q=0, 5$

   $MU' = 5 - 2Q$, then At $Q=0$, $MU'=5>0$ (Minimum)

   At $Q=5$, $MU' = -5 < 0$ (Maximum)

   Then, maximum utility of this consumer is at quantity=5.

3. a) $Q' = 12L - 0.6L^2 = 0$

   $0.6L(20 - L) = 0$, then $L=0, 20$

   $Q'' = 12 - 1.2L$; At $L=0$, $Q'' = 12>0$ (Minimum)

   At $L=20$, $Q'' = -12<0$ (Maximum)

   Then, L=20 is the level of labors that maximizes the output.

   b) Let AP= Average Product= $6L - 0.2L^2$

   $AP' = 6 - 0.4L = 0$, then $L=15$

   $AP'' = -0.4 < 0$ (Maximum)

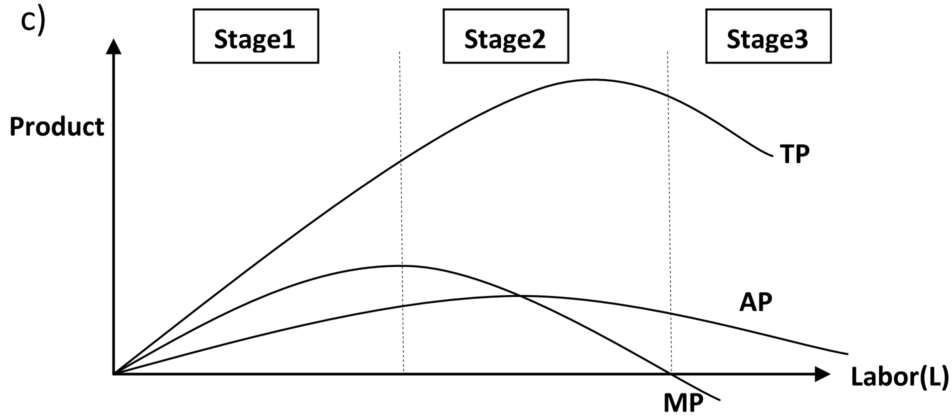
   Then, at L=15 gives maximum average product=$6(15) - 0.2(15)^2 = 45$

   And marginal product= $\frac{dQ}{dL} = 12L - 0.6L^2$

   Then, at L=15 gives marginal product= $12(15) - 0.6(15)^2 = 45$

It shows that when average product is maximized (at L=15), average product is equal to marginal product.

c)



Stage 2 is efficient in production. At the level of labor from where MP is maximized to MP is 0.

MP is maximized; MP' = $\frac{dMP}{dL}$ = 12 – 1.2L=0, then L=10, MP'' = -1.2 <0 (Maximum)

MP is 0, $12L - 0.6L^2=0$, then L=20

So, the range of level of labor from L=10 to L=20 represents as efficient production.

4. a) Total Variable Cost = $\frac{Q}{20}^3 - \frac{3Q^2}{10} + 2Q$

Total Fixed Cost= 4

Average Variable Cost= $\frac{Q}{20}^2 - \frac{3Q}{10} + 2$

Average fixed cost = $\frac{4}{Q}$

Average cost=$\frac{Q}{20}^2 - \frac{3Q}{10} + 2 + \frac{4}{Q}$

Marginal cost = $\frac{3Q^2}{20} - \frac{3Q}{5} + 2$

b) $AVC = \frac{Q}{20}^2 - \frac{3Q}{10} + 2$

$AVC' = \frac{Q}{10} - \frac{3}{10} = 0$, then Q=3

$AVC'' = \frac{1}{10} > 0$ (Minimum)

Q=3 is the output level that minimizes short-run average variable cost which is

$AVC = \frac{3}{20}^2 - \frac{3(3)}{10} + 2 = 15.5$

c) At Q=3, it gives minimal short run average cost= $AVC = \frac{3}{20}^2 - \frac{3(3)}{10} + 2$ =15.5

At Q=3, it gives marginal cost= $\frac{3(3)^2}{20} - \frac{3(3)}{5} + 2 = 15.5$

Then, at Q=3 which gives minimal average variable cost, average variable cost is equal to marginal cost.

5. a) $TC = Q^3 - 12Q^2 + 60Q + 10$, then $MC = 3Q^2 - 24Q + 60$
To maximize profit; P=MC, then $39 = 3Q^2 - 24Q + 60$
Q= 1, 7 (But $\frac{dMC}{dQ} > 0$) , so Q=7 only
Total Profit = $39(7) - (7^3 - 12(7)^2 + 60(7) + 10) = 88$

b) Shut down price is at minimum of AVC

$AVC = Q^2 - 12Q + 60$

$AVC' = 2Q - 12 = 0$, then Q=6

$AVC'' = 2 > 0$ (Minimum)

Then, at Q=6 , $AVC = 6^2 - 12(6) + 60 = 24$, so 24 is the shut-down price of the firm.

6. $TR = 50Q + 0.5Q^2$, so $MR = 50 + Q$

$TC=Q^3 - 8Q^2+60Q+2$, so $MC= 3Q^2 - 16Q+60$

To maximize profit; $MR=MC$

$$50+Q = 3Q^2 - 16Q+60$$

$$Q= 5, \frac{2}{3}$$

$$\frac{d(MR - MC)}{dQ} = \frac{d(-3Q^2+17Q-10)}{dQ} = -6Q+17$$

At Q= 5, $\frac{d(MR - MC)}{dQ} = -6(5)+17 = -13 <0$ (Maximum)

At Q= $\frac{2}{3}$, $\frac{d(MR - MC)}{dQ} = -6(\frac{2}{3})+17 =13 >0$ (Minimum)

Then, monopolist maximizes profit at Q=5 where price= $50+0.5(5)=52.5$

Monopolist profit= TR-TC= $50(5)+0.5(5)^2- ((5)^3 - 8(5)^2+60(5)+2) = 35.5$

And shut-down price is at minimum of AVC

$AVC= Q^2 - 8Q+60$, $AVC' =2Q -8 =0$, then Q=4

$AVC'' =2 >0$ (Minimum)

At Q=4, P=50+0.5(4)=52, so 52 is the shut-down price of the monopolist.

7. a) $Q=100 - P$, then $P= 100 - Q$, so $TR= 100Q - Q^2$

   b) Profit maximization is where MR=MC

   $TR=100Q - Q^2$, so $MR= 100 - 2Q$

   $C= \frac{1}{3} Q^3 - 7Q^2 +11Q +50$, so $MC=Q^2 - 14Q+11$

   $$MR=MC$$

   $$100 - 2Q= Q^2 - 14Q+11$$

   $$Q^2 - 12Q - 89 =0$$

8. a)$MC=3Q^2 - 12Q+19$

   $TC= \int(3Q^2 - 12Q+19)dQ = Q^3 - 6Q^2 +19Q +C$

   $AC=Q^2 - 6Q+19$

   $AC'=2Q - 6 =0$, then Q=3

   $AC''=2 >0$ (Minimum)

   Then, at Q=3 gives minimal average variable cost.

   b) $TR=34Q$ and $TC= Q^3 - 6Q^2 +19Q +50$

   Total Profit=34 - $[Q^3 - 6Q^2 +19Q +50] = -Q^3 +6Q^2 +15Q - 50$

Total Profit' = $-3Q^2+12Q+15=-Q^2+4Q+5=0$

Then, Q=5, -1 (But Q can't be negative)

Total Profit'' = $-6Q+12$

At Q=5, Total Profit'' = $-18<0$ (Maximum)

Total Profit of the firm= $-(5)^3+6(5)^2+15(5)-50=50$

9. a) $F= -2x^2 -2xy -2y^2+36x+42y - 158$

   2 conditions needed

   1. FOC: $F_x= -4x-2y+36=0$ -----1)

      $F_y= -2x-4y+42=0$ -----2)

      Solve 1) and 2), x=5, y=8

   2. SOC: $[H]_{2x2} = \begin{bmatrix} Fxx & Fxy \\ Fyx & Fyy \end{bmatrix} = \begin{bmatrix} -4 & -2 \\ -2 & -4 \end{bmatrix}$

      $[H_1] = |-4| = -4 < 0$

      $[H_2] = \begin{vmatrix} -4 & -2 \\ -2 & -4 \end{vmatrix} = 12 > 0$

      Then, x=5 and y=8 gives F maximum.

b) $F=8x^3+2xy-3x^2+y^2+1$

   2 conditions needed

   1.FOC: $F_x= 24x^2+2y-6x=0$ -----1)

   $F_y= 2x+2y=0$ -----2)

   Solve 1) and 2), then , $x_1=0$, $y_1=0$ and $x_2=\frac{1}{3}$, $y_2=-\frac{1}{3}$

   2.SOC: $[H]_{2x2} = \begin{bmatrix} Fxx & Fxy \\ Fyx & Fyy \end{bmatrix} = \begin{bmatrix} 48x-6 & 2 \\ 2 & 2 \end{bmatrix}$

   1st Case $x_1=0$, $y_1=0$

   $[H_1] = |-6| = -6 < 0$

   $[H_2] = \begin{vmatrix} -6 & 2 \\ 2 & 2 \end{vmatrix} = -16 < 0$

   Then, $x_1=0$, $y_1=0$ gives inflection point.

   2nd Case $x_2=\frac{1}{3}$, $y_2=-\frac{1}{3}$

   $[H_1] = |10| = 10 > 0$

   $[H_2] = \begin{vmatrix} 10 & 2 \\ 2 & 2 \end{vmatrix} = 16 > 0$

Then, $x_2 = \frac{1}{3}$, $y_2 = \frac{-1}{3}$ gives F minimum.

c) $F = xz + x^2 - y + yz + y^2 + 3z^2$

2 conditions needed

<u>1.FOC</u>: $F_x = z + 2x = 0$ -----1)

$F_y = -1 + z + 2y = 0$ -----2)

$F_z = x + y + 6z = 0$ -----3)

Solve 1), 2)and 3), then $x = \frac{1}{20}$, $y = \frac{11}{20}$, $z = \frac{-1}{10}$

<u>2.SOC:</u> $[H]_{3x3} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 6 \end{bmatrix}$

$[H_1] = |\ 2\ | = 2 > 0$

$[H_2] = \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4 > 0$

$[H_3] = \begin{vmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 6 \end{vmatrix} = 20 > 0$

Then, $x = \frac{1}{20}$, $y = \frac{11}{20}$, $z = \frac{-1}{10}$ gives F minimum.

d) $F = 2x^2 + xy + 4y^2 + xz + z^2 + 3$

2 conditions needed

<u>1.FOC</u>: $F_x = 4x + y + z = 0$ -----1)

$F_y = x + 8y = 0$ -----2)

$F_z = x + 2z = 0$ -----3)

Solve 1), 2)and 3), then x= 0 , y=0 , z=0

<u>2.SOC:</u> $[H]_{3x3} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 8 & 0 \\ 1 & 0 & 2 \end{bmatrix}$

$[H_1] = |\ 1\ | = 1 > 0$

$[H_2] = \begin{vmatrix} 1 & 1 \\ 1 & 8 \end{vmatrix} = 7 > 0$

$$[H_3] = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 8 & 0 \\ 1 & 0 & 2 \end{vmatrix} = 6 > 0$$

Then, x= 0 , y=0 , z=0gives F minimum.

10. $F=4xy+12x+16y-2x^2-3y^2$

2conditions needed

<u>1.FOC</u>: $F_x= 4y+12 - 4x=0$ -----1)

$F_y= 4x+16 - 6y =0$ -----2)

Solve 1) and 2), x=17, y=14

<u>2.SOC:</u> $[H]_{2x2} = \begin{bmatrix} Fxx & Fxy \\ Fyx & Fyy \end{bmatrix} = \begin{bmatrix} -4 & 4 \\ 4 & -6 \end{bmatrix}$

$[H_1] = | -4 | = -4 < 0$

$[H_2] = \begin{vmatrix} -4 & 4 \\ 4 & -6 \end{vmatrix} = 8 > 0$

x=17and y=14 gives F maximum,

So, 17 kilograms of fertilizer x and 14 kilograms of fertilizer y are used to yield maximum F.

11. $TR= 70Q_1+50Q_2$ and $TC= Q_1^2+Q_1Q_2+Q_2^2$

So, total profit = TR – TC = = $70Q_1+50Q_2- Q_1^2 - Q_1Q_2 -Q_2^2$

2conditions needed

<u>1.FOC</u>: $F_1= 70 - 2Q_1 - Q_2=0$ -----1)

$F_2= 50 - Q_1 - 2Q_2=0$ -----2)

Solve 1) and 2), $Q_1=30$, $Q_2=10$

<u>2.SOC:</u> $[H]_{2x2} = = \begin{bmatrix} -2 & -1 \\ -1 & -2 \end{bmatrix}$

$[H_1] = | -2 | = -2 < 0$

$[H_2] = \begin{vmatrix} -2 & -1 \\ -1 & -2 \end{vmatrix} = 3 > 0$

Then, the firm should produce 2 goods as $Q_1=30$, $Q_2=10$ which give maximum profit to the firm.

Total Profit = $70(30)+50(10)- 30^2 -30(10) -10^2 =1300$

12. $Q_1=40 - 2P_1 + P_2$ -----1)

$Q_2=15+P_1 - P_2$ -----2)

1)+2) $Q_1+Q_2 = 55 - P_1$

$P_1= 55 - Q_1 - Q_2$, $TR_1= 55Q_1 - Q_1^2 - Q_1Q_2$

2x2) +1) $Q_1+2Q_2=70 - P_1$

$P_2=70 - Q_1 - 2Q_2$, $TR_2 = 70Q_2 - Q_1Q_2 - 2Q_2^2$

Total Profit= $TR_1+TR_2 - TC$

$$=55Q_1+70Q_2 - Q_1^2 - 2Q_1Q_2 - 2Q_2^2 - Q_1^2 - Q_1Q_2 - Q_2^2$$

$$=55Q_1+70Q_2 - 3Q_1Q_2 - 3Q_2^2 - 2Q_1^2$$

<u>1.FOC:</u> Total $Profit_1= 55 - 3Q_2 - 4Q_1 = 0$ -----3)

Total $Profit_2=70 - 3Q_1 - 6Q_2 = 0$ -----4)

Solve 3) and 4), $Q_1=8$ and $Q_2=\frac{23}{3}$

<u>2.SOC</u>: $[H] = \begin{bmatrix} -4 & -3 \\ -3 & -6 \end{bmatrix}$

$|H_1|= -4 <0$

$|H_2| = \begin{vmatrix} -4 & -3 \\ -3 & -6 \end{vmatrix} = 15>0$

Then, to maximize profit, the firm should produce at $Q_1=8$ and $Q_2=\frac{23}{3}$ with total profit=488.

13.a) $TR= 1000(Q_1+Q_2)$, $TC= 500Q_1+2Q_1^2+100Q_2+Q_2^2$

So, total profit= $1000(Q_1+Q_2) - (500Q_1+2Q_1^2+100Q_2+Q_2^2)=500Q_1+900Q_2 - 2Q_1^2 - Q_2^2$

To maximize total profit; 2 conditions needed

<u>1.FOC:</u> Total $Profit_1= 500 - 4Q_1 = 0$ -----1)

Total $Profit_2=900 - 2Q_2 = 0$ -----2)

Solve 1) and 2), $Q_1=125$ and $Q_2= 450$

$\underline{2.SOC}$: $[H] = \begin{bmatrix} -4 & 0 \\ 0 & -2 \end{bmatrix}$

$|H_1| = -4 < 0$

$|H_2| = \begin{vmatrix} -4 & 0 \\ 0 & -2 \end{vmatrix} = 8 > 0$

Then, to maximize profit, the firm should produce at $Q_1=125$ and $Q_2= 450$.

b) If the producer owns the monopoly power;

$TR=1575Q - Q^2=1575(Q_1+Q_2) - (Q_1+Q_2)^2= 1575Q_1+1575Q_2 - Q_1^2 - Q_2^2 - 2Q_1Q_2$

$TC= 500Q_1+2Q_1^2+100Q_2+Q_2^2$

Total Profit= $TR - TC = 1075Q_1+1475Q_2 - 3Q_1^2 - 2Q_2^2 - 2Q_1Q_2$

To maximize total profit; 2 conditions needed

$\underline{1.FOC}$: Total $Profit_1= 1075- 6Q_1 - 2Q_2 = 0$ -----1)

Total $Profit_2=1475- 4Q_2 - 2Q_1 = 0$ -----2)

Solve 1) and 2), $Q_1=67.5$ and $Q_2= 335$

$\underline{2.SOC}$: $[H] = \begin{bmatrix} -6 & -2 \\ -2 & -4 \end{bmatrix}$

$|H_1| = -6 < 0$

$|H_2| = \begin{vmatrix} -6 & -2 \\ -2 & -4 \end{vmatrix} = 20 > 0$

Then, to maximize profit, the firm should produce at $Q_1=67.5$ and $Q_2= 335$.

14. $P_1 = 40-2Q_1$ and $P_2=90-4Q_2$

$TR_1=40Q_1 - 2Q_1^2$ and $TR_2=90Q_2 - 4Q_2^2$

Then, $TR= TR_1+TR_2 =40Q_1 - 2Q_1^2+90Q_2 - 4Q_2^2$ and $TC=10+18Q_1+18Q_2$

Total Profit= $40Q_1 - 2Q_1^2+90Q_2 - 4Q_2^2 - (10+18Q_1+18Q_2)$

$= 22Q_1+72Q_2 - 2Q_1^2 - 4Q_2^2 - 10$

To maximize profit; 2 conditions needed

1.FOC: Total $Profit_1$= 22– $4Q_1$ = 0 -----1)

Total $Profit_2$=72– $8Q_2$ = 0 -----2)

Solve 1) and 2), Q_1=5.5 and Q_2= 9

2.SOC: $[H] = \begin{bmatrix} -4 & 0 \\ 0 & -8 \end{bmatrix}$

$|H_1|$= - 4 <0

$|H_2| = \begin{vmatrix} -4 & 0 \\ 0 & -8 \end{vmatrix} = 32 > 0$

Then, to maximize profit, the firm should produce at Q_1=5.5 and Q_2= 9.

And total profit=22(5.5)+72(9) – $2(5.5)^2$– $4(9)^2$ – 10= 374.5

15.Q=$5K^{1/2}L^{1/4}$

TR=4($5K^{1/2}L^{1/4}$) =20 $K^{1/2}L^{1/4}$

TC=5K+10L

Total Profit= 20 $K^{1/2}L^{1/4}$– 5K – 10L

To maximize profit; 2 conditions needed

1.FOC: Total $Profit_K$= 10 $K^{-1/2}L^{1/4}$ – 5 = 0 -----1)

Total $Profit_L$=5 $K^{1/2}L^{-3/4}$ – 10 = 0 -----2)

Solve 1) and 2), L=1 and K=4

2.SOC: $[H] = \begin{bmatrix} -5L^{1/4}K^{-3/2} & \frac{5}{2}K^{-1/2}L^{-3/4} \\ \frac{5}{2}L^{-3/4}K^{-1/2} & -\frac{15}{4}K^{1/2}L^{-7/4} \end{bmatrix} = \begin{bmatrix} -\frac{5}{8} & \frac{5}{4} \\ \frac{5}{4} & \frac{-15}{2} \end{bmatrix}$

$|H_1| = -\frac{5}{8} < 0$

$|H_2| = \begin{vmatrix} -\frac{5}{8} & \frac{5}{4} \\ \frac{5}{4} & \frac{-15}{2} \end{vmatrix} = \frac{25}{8} > 0$

Then, to maximize profit, the firm should use labor and capital as L=1 and K=4.

16.a) Price discrimination

$Q_1 = 24 - 0.2P_1$, so $P_1=120 - 5Q_1$ and $TR_1=120Q_1 - 5Q_1^2$

$Q_2 = 10 - 0.05P_2$, so $P_2=200 - 20Q_2$ and $TR_2=200Q_2 - 20Q_2^2$

$TC = 35+40Q= 35+40 (Q_1+Q_2)$

Total Profit= $TR_1+TR_2 - TC$

$=120Q_1 - 5Q_1^2+200Q_2 - 20Q_2^2 - [35+40 (Q_1+Q_2)]$

$=80Q_1 + 160Q_2 - 5Q_1^2 - 20Q_2^2 - 35$

To maximize profit; 2 conditions needed

1.FOC: Total $\text{Profit}_1= 80 - 10Q_1 = 0$ -----1)

Total $\text{Profit}_2=160 - 40Q_2 = 0$ -----2)

Solve 1) and 2), $Q_1=8$ and $Q_2= 4$

2.SOC: $[H] = \begin{bmatrix} -10 & 0 \\ 0 & -40 \end{bmatrix}$

$|H_1| = -10 < 0$

$|H_2| = \begin{vmatrix} -10 & 0 \\ 0 & -40 \end{vmatrix} = 400 > 0$

Then, to maximize profit, the firm should produce at $Q_1=8$ and $Q_2= 4$.

b) Non-price discrimination

As $P_1=P_2$, from $Q_1 = 24 - 0.2P_1$ and $Q_2 = 10 - 0.05P_2$

So, $Q=Q_1+Q_2=24 - 0.2P_1+10 - 0.05P_2 = 34 - 0.25P$

$P=136 - 4Q$, then $TR=136Q - 4Q^2$

$TC=35+40Q$

Total Profit = $136Q - 4Q^2 - 35 - 40Q = 96Q - 4Q^2 - 35$

To maximize profit; 2 conditions needed

1.FOC: Total Profit' = $\frac{d(Total\ \ Profit\ )}{dQ}$= 96 – 8Q=0

Q=12, so Q_1=6 and Q_2=6

2.SOC: Total profit'' = - 8 <0 (Maximum)

Then, to maximize profit, if non-price discrimination, the firm should produce at Q_1=6 and Q_2= 6.

c) Total profit if price discrimination= 80(8) + 160(4)– $5(8)^2$– $20(4)^2$ – 35=605

Total profit if non-price discrimination= 96(12) – $4(12)^2$ – 35=541

So, the firm can gain higher profit if they use the strategy of price discrimination.