B.E. International Program

Faculty of Economics, Thammasat University

EE 320 Introductory Mathematical Economics (Section 046402)

Semester 1/2013

Practice Problem 8
(Optimization without Constraint: Multiple-Variable Cases) [1]

1. Yearly profits for a firm are given by

$$P(x,y) = -x^2 - y^2 + 22x + 18y - 102$$

where x is the amount spent on research, and y is the amount spent on advertising.

a) Find the profits when x = 10, y = 8, and when x = 12, y = 10.

b) Find the only possible values of x and y that can maximize profits, and the corresponding profit.

2. A firm produces two goods. The cost of producing x unit of good 1 and y units of good 2 is

$$C(x,y) = x^2 + xy + y^2 + x + y + 14$$

Suppose that the firm sells all its output of each good at prices per unit p and q respectively. Find the values of x and y that maximize profits. (Assume $\frac{1}{2}p + \frac{1}{2} < q < 2p - 1$ and $p > 1$).

3. The profit function of a firm is $\pi(x,y) = px + qy - \alpha x^2 - \beta y^2$, where p and q are the prices per unit and $\alpha x^2 + \beta y^2$ are the costs of producing and selling x units of the first good and y units of the other. The constants are all positive.

[1] Questions 1-9 are from Sydsaeter and Hammond, 2008. Questions 10-11 are from Wainwright.

a) Find the values of x* and y* that maximize profits. Verify that the second-order conditions are satisfied.

b) Define $\pi^*(p,q) = \pi(x^*, y^*)$ Verify that $\frac{\partial \pi^*(p,q)}{\partial p} = x^*$ and $\frac{\partial \pi^*(p,q)}{\partial q} = y^*$ Give these results economic interpretations.

4. Each of two firms A and B produces its own brand of a commodity, such as mineral water, in amounts denoted by x and y, and these are sold at prices p and q unit, respectively. Each firm determines its own price and produces exactly as much as is demanded. The demands for the two brands are given by

$$x = 29 - 5p + 4q$$

$$y = 16 + 4p - 6q$$

Firm A has total costs $5 + x$, whereas firm B has total costs $3 + 2y$. (Assume that the functions to be maximized have maxima, and at positive prices.)

a) Initially, the two firms collude in order to maximize their combined profits, as one monopolist would. Find the prices (p, q), the production levels (x, y), and the profits of firms A and B.

b) Then, an antitrust authority prohibits collusion, so each producer maximizes its own profit, taking the other's price as given.

If q is fixed, how will A choose p? (Find p as a function $p = p_A(q)$.)

If p is fixed, how will B choose q? (Find q as a function $q = q_B(p)$.)

c) Under the assumptions in part b), what constant equilibrium prices are possible? What are the production levels and profits in this case?

5. A profit-maximizing monopolist produces two commodities whose quantities are denoted by x_1 and x_2. Good 1 is subsidized at the rate of *s* per unit and good 2 is taxed at *t* per unit. The monopolist's profit function is therefore given by

$$\pi(x_1, x_2) = R(x_1, x_2) - C(x_1, x_2) + Sx_1 - tx_2$$

where R and C are the firm's revenue and cost functions, respectively. Assume that the partial derivatives of these functions have the following signs:

$$R_1' > 0,\ \ R_2' > 0,\ \ R_{11}' < 0,\ \ R_{22}' < 0,\ \ R_{12}' = R_{21}' < 0$$

$$C_1' > 0,\ \ C_2' > 0,\ \ C_{11}' > 0,\ \ C_{22}' > 0,\ \ C_{12}' = C_{21}' > 0$$

a) Find the first-order conditions for maximum profits.

b) Write down the second-order sufficient conditions for maximum profit.

c) Suppose that $x_1^* = x_1^*(s,t)$, $x_2^* = x_2^*(s,t)$ solve the problem. Find the signs of $\partial x_1^*/\partial s$, $\partial x_1^*/\partial t$, $\partial x_2^*/\partial s$, $\partial x_2^*/\partial t$, assuming that the SOSC's are satisfied.

d) Show that $\partial x_1^*/\partial t = -\,\partial x_2^*/\partial s$.

6. A firm produces two different kinds A and B of a commodity. The daily cost of producing Q_1 units of A and Q_2 units of B is $C(Q_1, Q_2) = 0.1(Q_1^2 + Q_1Q_2 + Q_2^2)$. Suppose that the firm sells all its output at a price per unit $P_1 = 120$ for A and $P_2 = 90$ for B.

a) Find the daily production levels that maximize profit.

b) What prices (P_1) per unit of A would imply that the optimal daily production level for A is 400 units?

7. Define *f(x, y)* for all *(x, y)* by

$$f(x,y) = e^{x+y} + e^{x-y} - \frac{3}{2}x - \frac{1}{2}y$$

a) Find the first- and second-order partial derivatives of *f*, then show that *f(x, y)* is convex.

b) Find the minimum point of *f(x, y).*

8.

a) Find and classify the stationary points of

$$f(x,y) = x^2 - y^2 - xy - x^3$$

b) Find the domain *S* where *f* is concave, and find the largest value *f* in *S*.

9. Consider a firm that produces two different goods A and B. If the total cost function is *C(x, y)* and the prices obtained per unit of A and B are *p* and *q* respectively, then the profit is

$$\pi(x,y) = px + qy - C(x,y) \qquad \text{(i)}$$

a) Suppose first that the firm has small shares in the markets for these goods and takes p and q as given. Write down and interpret the first-order conditions for $x^* > 0$ and $y^* > 0$ to maximize profits.

b) Suppose next that the firm has a monopoly in the sale of both goods. The prices are no longer fixed, but chosen by the monopolist, bearing in mind the demand functions

$$x = f(p,q) \quad \text{and} \quad y = g(p,q) \qquad \text{(ii)}$$

Then profit as a function of p and q is

$$\hat{\pi}(p,q) = pf(p,q) + qg(p,q) - C(f(p,q), g(p,q)) \qquad \text{(iii)}$$

Write down the first-order conditions for $p^* > 0$ and $q^* > 0$ to maximize profits.

c) Suppose $x = a - bp + cq$ and $y = \alpha + \beta p - \gamma q$, where b and γ are positive. (An increase in the price of either good lowers the demand for that good. But an increase in the price of one good may increase or decrease the demand for the other.) If the cost function is $C(x,y) = Px + Qy + R$, write down the first-order conditions for maximum-profit.

d) Prove that the second-order conditions are satisfied provided $4\gamma b \geq (\beta + c)^2$.

e) Suppose instead we solve equations (ii) in part (b) for p and q to obtain the inverse demand functions

$$p = F(x,y) \quad \text{and} \quad q = G(x,y) \qquad \text{(iv)}$$

Then the profit as a function of x and y is

$$\pi(x,y) = xF(x,y) + yG(x,y) - C(x,y) \qquad \text{(v)}$$

Write down the first-order conditions for $x^* > 0$ and $y^* > 0$ to maximize profits. (Of course, if the correspondence between (x, y) and (p, q) defined by the problem in part b is unique, the optimal choices of x, y, p, and q are the same.)

f) Suppose $p = a - bx - cy$ and $q = \alpha - \beta x - \gamma y$, where b and γ are positive. (An increase in the price of either good lowers the demand for that good. But an increase in the price of one good may increase or decrease the demand for the other.) If the cost function is $C(x,y) = Px + Qy + R$, write down the first-order conditions for maximum-profit.

g) Prove that the second-order conditions are satisfied provided $4\gamma b \geq (\beta + c)^2$.