Solution to past exam Dec 2015

1.

(1) $\hat{\underline{b}} = \dfrac{\underline{u}\,\underline{u}^T}{\underline{u}^T\underline{u}}\,\underline{b}$ (since col $\underline{A}$ = span $\left\{\begin{bmatrix}3\\1\end{bmatrix}\right\}$)

Projection of $\underline{b}$ onto col $\underline{A}$

a) $\underline{P}_c = \dfrac{\underline{u}\,\underline{u}^T}{\underline{u}^T\underline{u}} = \dfrac{\begin{bmatrix}3\\1\end{bmatrix}\begin{bmatrix}3 & 1\end{bmatrix}}{10} = \dfrac{1}{10}\begin{bmatrix}9 & 3\\3 & 1\end{bmatrix}$

b) Row $\underline{A}$ = span $\left\{\begin{bmatrix}1\\1\\2\end{bmatrix}\right\}$ =

$$\underline{P}_R = \frac{\begin{bmatrix}1\\1\\2\end{bmatrix}\begin{bmatrix}1 & 1 & 2\end{bmatrix}}{6} = \frac{1}{6}\begin{bmatrix}1 & 1 & 2\\1 & 1 & 2\\2 & 2 & 4\end{bmatrix}$$

Closest distance $\hat{\underline{b}} = \underline{P}_R\,\underline{b} = \dfrac{1}{6}\begin{bmatrix}1\\1\\2\end{bmatrix}$

c) $\underline{P}_C\underline{A} = \dfrac{1}{10}\begin{bmatrix}9 & 3\\3 & 1\end{bmatrix}\begin{bmatrix}3 & 3 & 6\\1 & 1 & 2\end{bmatrix} = \begin{bmatrix}3 & 3 & 6\\1 & 1 & 2\end{bmatrix} = \underline{A}$

$\underline{P}_C\underline{A}\underline{P}_R = \begin{bmatrix}3 & 3 & 6\\1 & 1 & 2\end{bmatrix}\dfrac{1}{6}\begin{bmatrix}1 & 1 & 2\\1 & 1 & 2\\2 & 2 & 4\end{bmatrix} = \begin{bmatrix}3 & 3 & 6\\1 & 1 & 2\end{bmatrix} = \underline{A}$

The two multiplications project the columns/rows of $\underline{A}$ onto the column/row space of A.
This does not change the matrix.

d) basis set = $\begin{bmatrix}-1\\1\\0\end{bmatrix}, \begin{bmatrix}-2\\0\\1\end{bmatrix}$

② $(A^TA)\hat{x} = \underline{A}^T\underline{b}$

It is noted that this case (A^TA) has no inverse, therefore there is no unique least sq. solution

$\therefore \left[\underline{A}^T\underline{A} \mid \underline{A}^T\underline{B}\right] \sim \cdots \left[\begin{array}{ccc|c} 1 & 0 & 1 & 5 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$$\therefore \boxed{\hat{x} = \begin{bmatrix} 5 \\ -3 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}}$$

③ $\underline{A}$ is symmetric because the eigenvalues are real and the eigenvectors are orthogonal.

- TRACE $= 0+1$
- $\det A = 0$

$$\underline{A} = \underline{P}\underline{D}\underline{P}^{-1} = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\frac{1}{5}\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$$

$$\underline{A} = \frac{1}{5}\begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix}$$

$$\boxed{\underline{A}^5 = \underline{P}\underline{D}^5\underline{P}^{-1} = \begin{bmatrix} 2/5 \\ -1/5 \end{bmatrix}}$$

4.

(4) MARKOV process

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_{k+1} = \begin{bmatrix} 0.5 & 0.5 & 0.5 \\ 0.25 & 0.5 & 0 \\ 0.25 & 0 & 0.5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_k$$

Markov matrix $\rightarrow \lambda_1 = 1$

$\underline{A}$ is singular. $\rightarrow \lambda_3 = 0$

$$\text{TRACE}(A) = \lambda_1 + \lambda_2 + \lambda_3 = a_{11} + a_{22} + a_{33}$$

$$= 1 + \lambda_2 + 0 = 0.5 + 0.5 + 0.5$$

$$\boxed{\therefore \lambda_2 = 0.5}$$

For $\lambda_1 = 1$;

$$\underline{v}_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

For $\lambda_2 = 0.5$, $\underline{v}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

For $\lambda_3 = 0$

$$\underline{v}_3 = \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix} \text{ OR } \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

(b) $\underline{u}_0 = \begin{bmatrix} 6 \\ 0 \\ 6 \end{bmatrix}$

$$u_k = \underline{A}^k \underline{u}_0 = \underline{P}\underline{D}^k\underline{P}^{-1} \underline{u}_0 = \boxed{\begin{bmatrix} 6 \\ 3 \\ 3 \end{bmatrix}}$$