Topic 5

Input and Production Functions (Chapter 6)

Key Definitions

Productive resources, such as labor and capital equipment, that firms use to manufacture goods and services are called inputs or factors of production.

The amount of goods and services produces by the firm is the firm's output.

Production transforms a set of inputs into a set of outputs

Technology determines the quantity of output that is feasible to attain for a given set of inputs.

Key Definitions

The production function tells us the *maximum* possible output that can be attained by the firm for any given quantity of inputs.

Production Function: $Q = f(L, K)$

- Q = output
- K = Capital
- L = Labor

The production set is a set of technically feasible combinations of inputs and outputs.

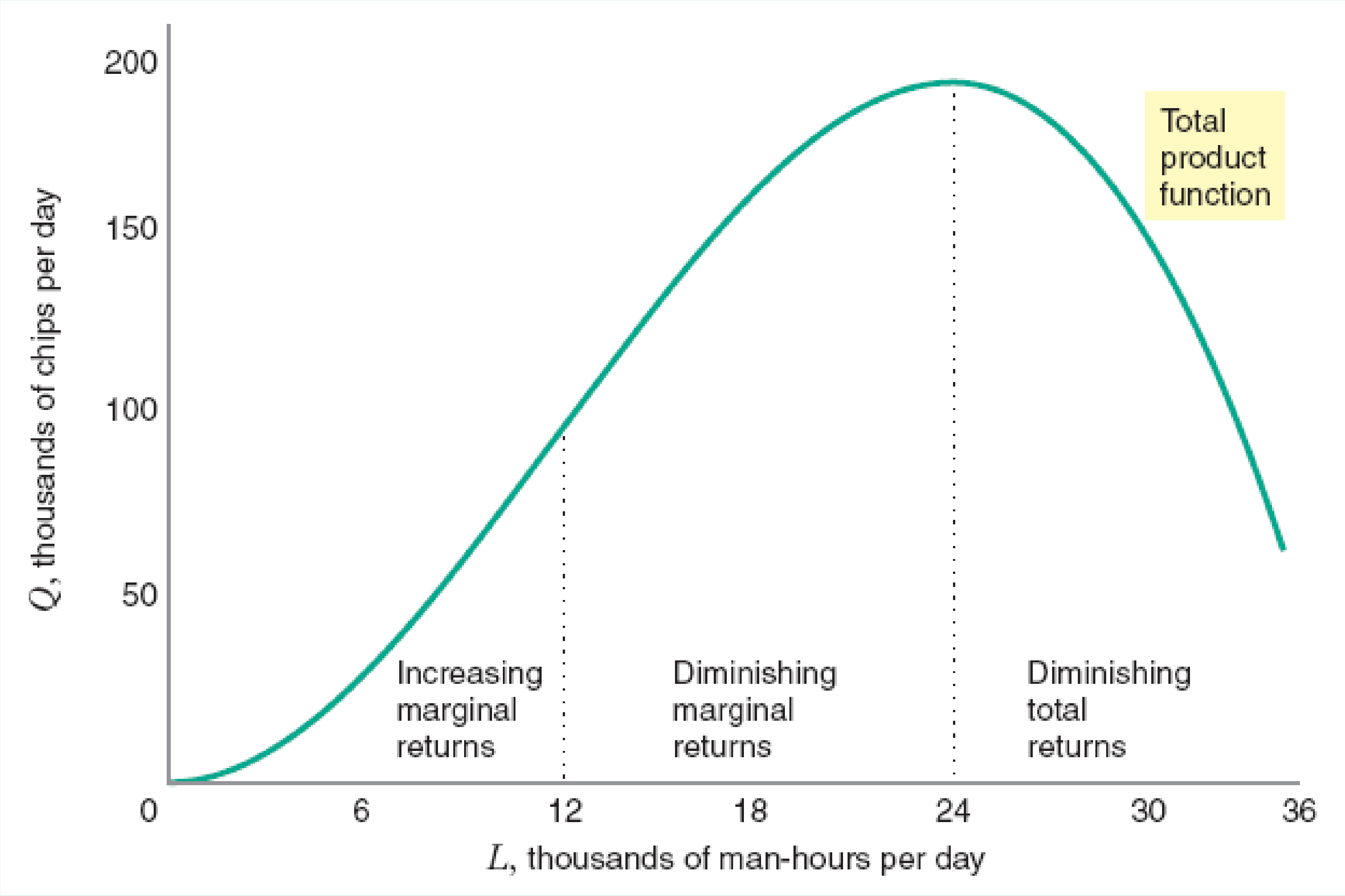
Total Product – Single Input (K is constant)

- **Total Product Function**: A single-input production function shows how total output depends on the level of the one input.

- **Increasing Marginal Returns to Labor**: An increase in the quantity of labor increases total output at an increasing rate.
- **Diminishing Marginal Returns to Labor**: An increase in the quantity of labor increases total output **but** at a decreasing rate.
- **Diminishing Total Returns to Labor**: An increase in the quantity of labor decreases total output.

Total Product – Single Input (K is constant)

Single-Input: L is variable while K is constant.



The Marginal Product

Definition: The **marginal product** of an input is the change in output that results from a small change in an input *holding the levels of all other inputs constant*.

$MP_L = \Delta Q/\Delta L$

- *(holding constant all other inputs)*

$MP_K = \Delta Q/\Delta K$

- *(holding constant all other inputs)*

Example: $Q = K^{1/2}L^{1/2}$

$MP_L = (1/2)L^{-1/2}K^{1/2}$

$MP_K = (1/2)K^{-1/2}L^{1/2}$

The Average Product & Diminishing Returns

Definition: The **average product** of an input is equal to the total output that is to be produced divided by the quantity of the input that is used in its production:

$$AP_L = Q/L \qquad AP_K = Q/K$$

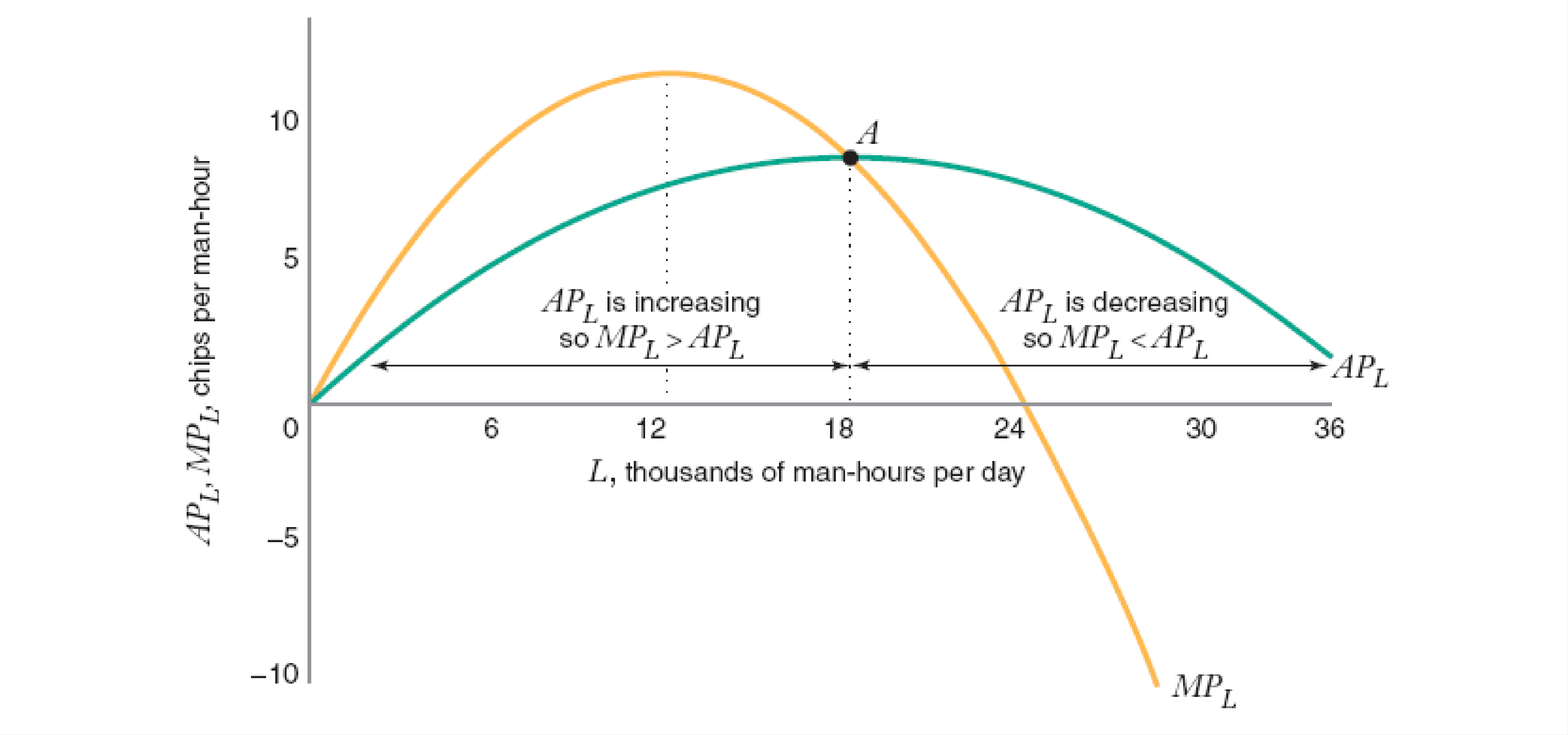
Example:

$AP_L = [K^{1/2}L^{1/2}]/L = K^{1/2}L^{-1/2}$ $\qquad$ $AP_K = [K^{1/2}L^{1/2}]/K = L^{1/2}K^{-1/2}$

Definition: The **law of diminishing marginal returns** states that marginal products *(eventually)* decline as the quantity used of a single input increases.

Total, Average, and Marginal Products

Single-Input: L is variable while K is constant.



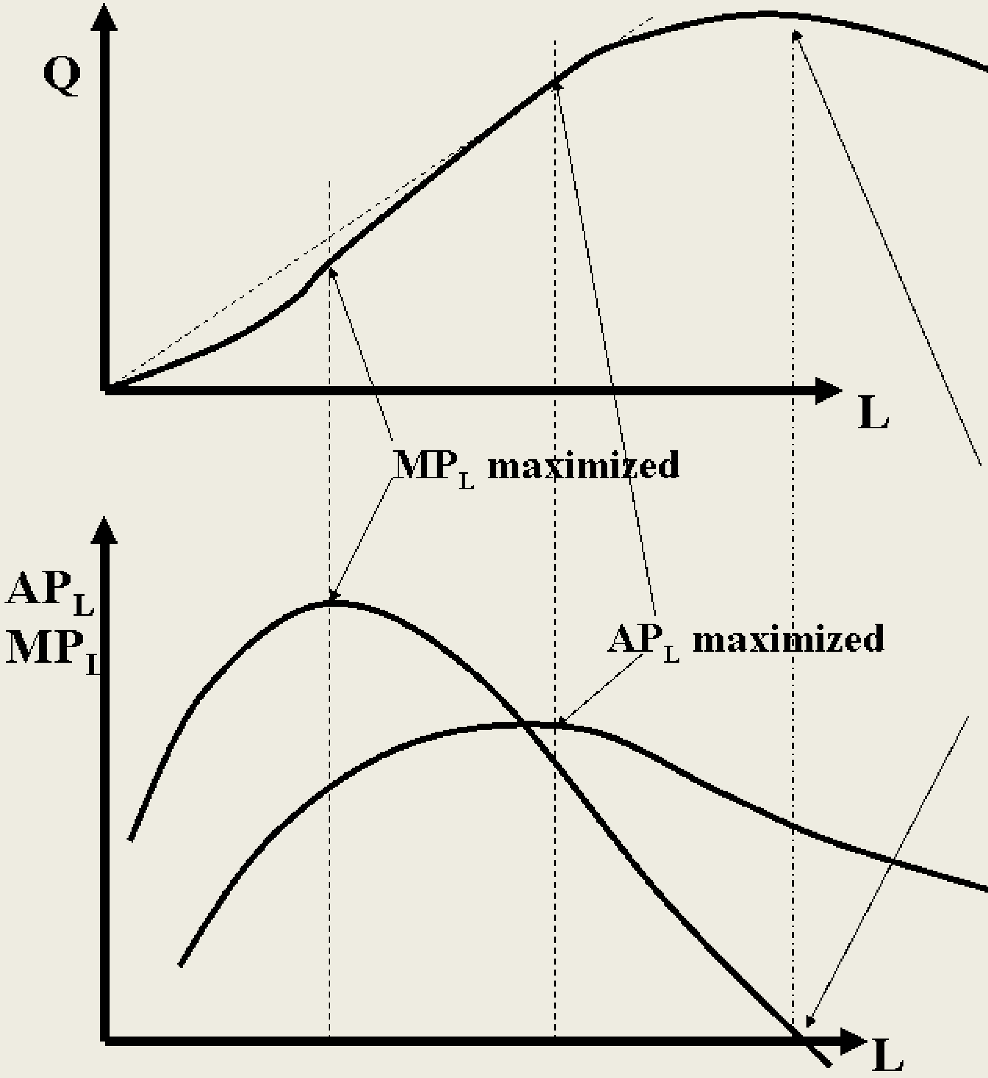
Total, Average, and Marginal Products

Single-Input: L is variable while K is constant.

- When AP is increasing in L, MPL > APL.
- When AP is decreasing in L, MPL < APL.
- When AP does not increase or decrease, MPL = APL.

Total, Average, and Marginal Products

Single-Input: L is variable while K is constant.



- TP_L is maximized where MP_L is zero.
- TP_L falls where MP_L is negative.
- TP_L rises where MP_L is positive.

Isoquants (for 2-input production)

Definition: An **isoquant** traces out all the combinations of inputs (labor and capital) that allow that firm to produce the same quantity of output

Example: $Q = K^{1/2}L^{1/2}$

*What is the **equation** of the isoquant for Q = 20?*

$20 = K^{1/2}L^{1/2}$

=> $400 = KL$

=> $K = 400/L$

Slope = dK/dL

Isoquants (for 2-input production)

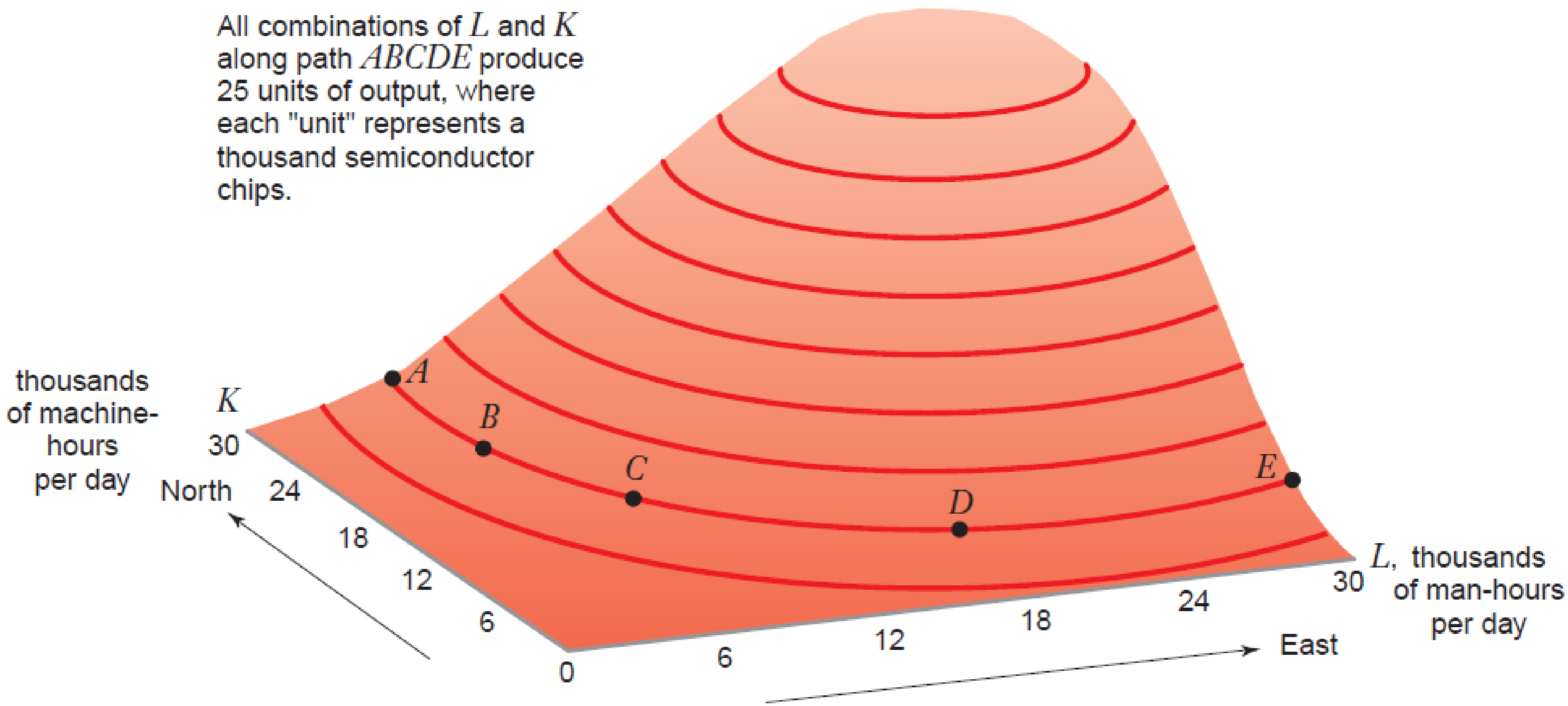
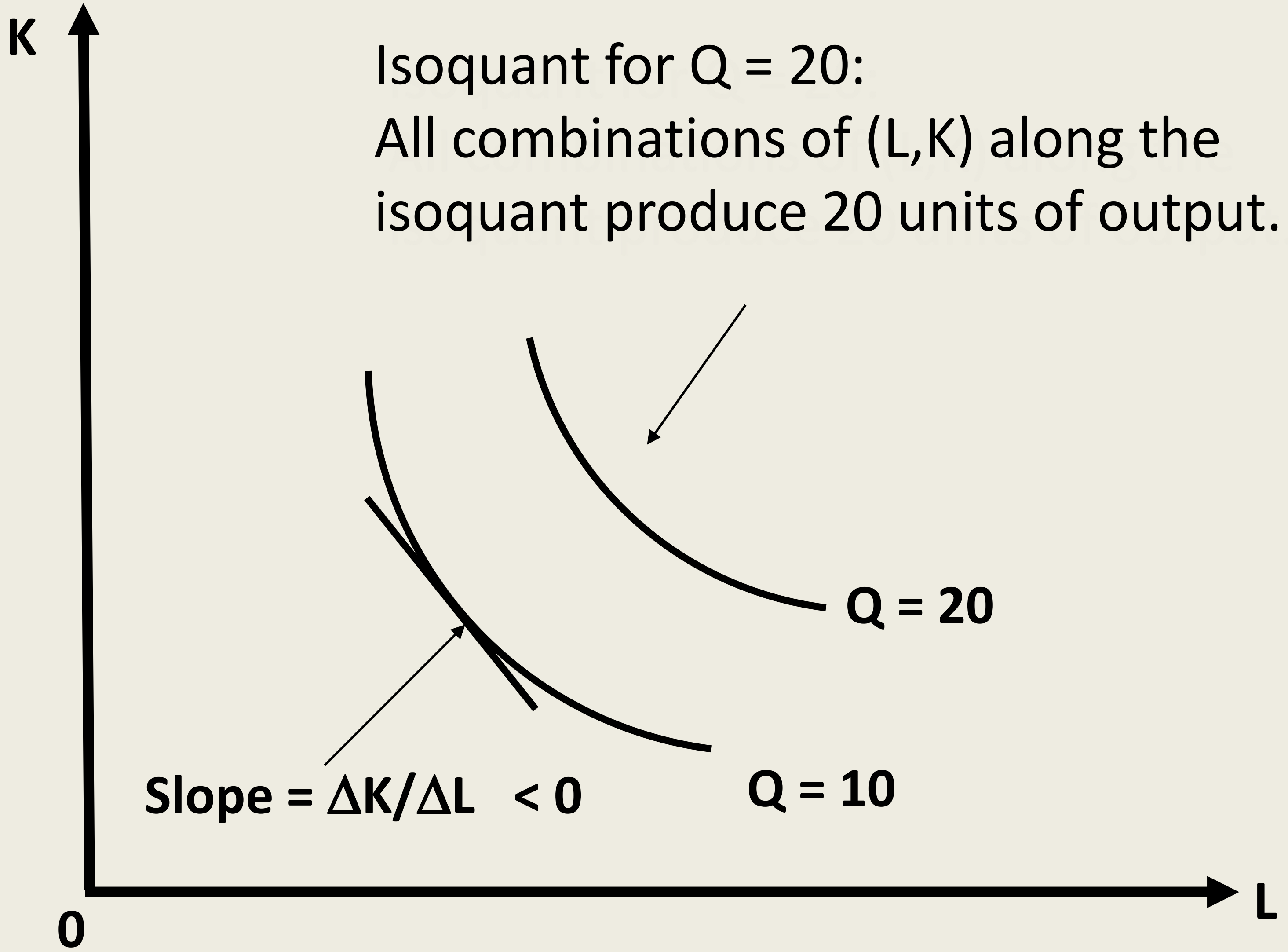


FIGURE 6.6 **Isoquants and the Total Product Hill**
If we start at point *A* and walk along the hill so that our elevation remains unchanged at 25 units of output, then we will trace out the path *ABCDE*. This is the 25-unit isoquant for this production function.

Isoquants (for 2-input production)



Marginal Rate of Technical Substitution

Definition: The **marginal rate of technical substitution (MRTS)** is the rate at which capital can be reduced for every one-unit increase in labor, holding the quantity of output constant.

$MRTS_{L,K} = -\Delta K/\Delta L$ *(for a constant level of output)*

Example

MRTS = 4

The firm can reduce capital by 4 units if it wants to employ one more labor. In doing so, the output will not change.

Marginal Rate of Technical Substitution

$MRTS_{L,K} = -\Delta K/\Delta L =$ **MPL/MPK**

<u>Proof</u>

$$\Rightarrow \frac{MP_L}{MP_K} = MRTS_{L,K}$$

Marginal Rate of Technical Substitution

Different ways to think about MRTS

- The rate at which the quantity of capital that can be *decreased* for every unit of *increase* in the quantity of labor, holding the quantity of output constant,

or

- The rate at which the quantity of capital that can be *increased* for every unit of *decrease* in the quantity of labor, holding the quantity of output constant

Marginal Rate of Technical Substitution

- If both marginal products are positive, the slope of the isoquant is negative.

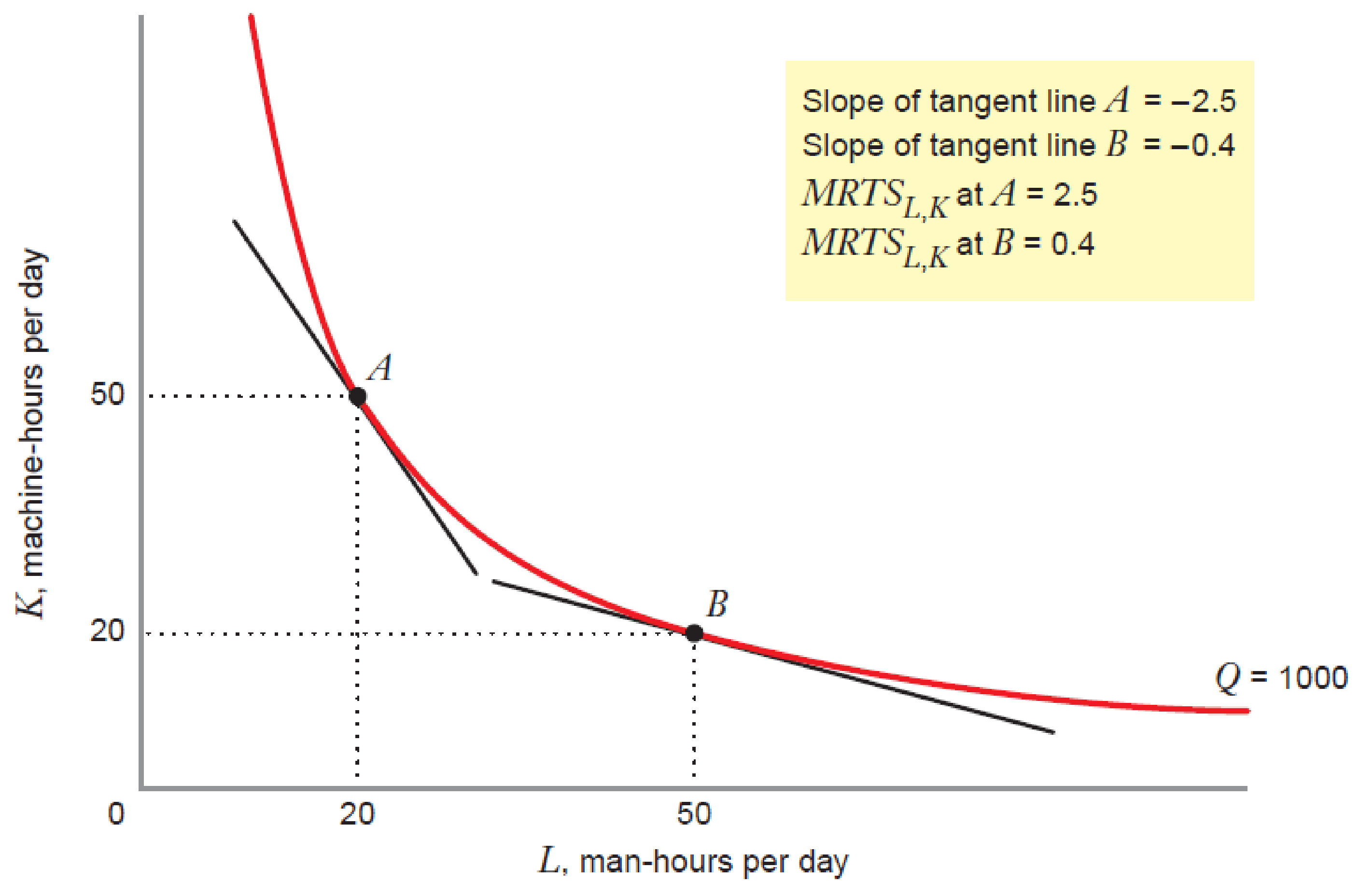
- If we have diminishing marginal returns, we also have a diminishing marginal rate of technical substitution - the marginal rate of technical substitution of labor for capital diminishes as the quantity of labor increases, along an isoquant – isoquants are convex to the origin.

- For many production functions, marginal products eventually become negative. **We don't graph parts of the isoquants where marginal products are negative.**

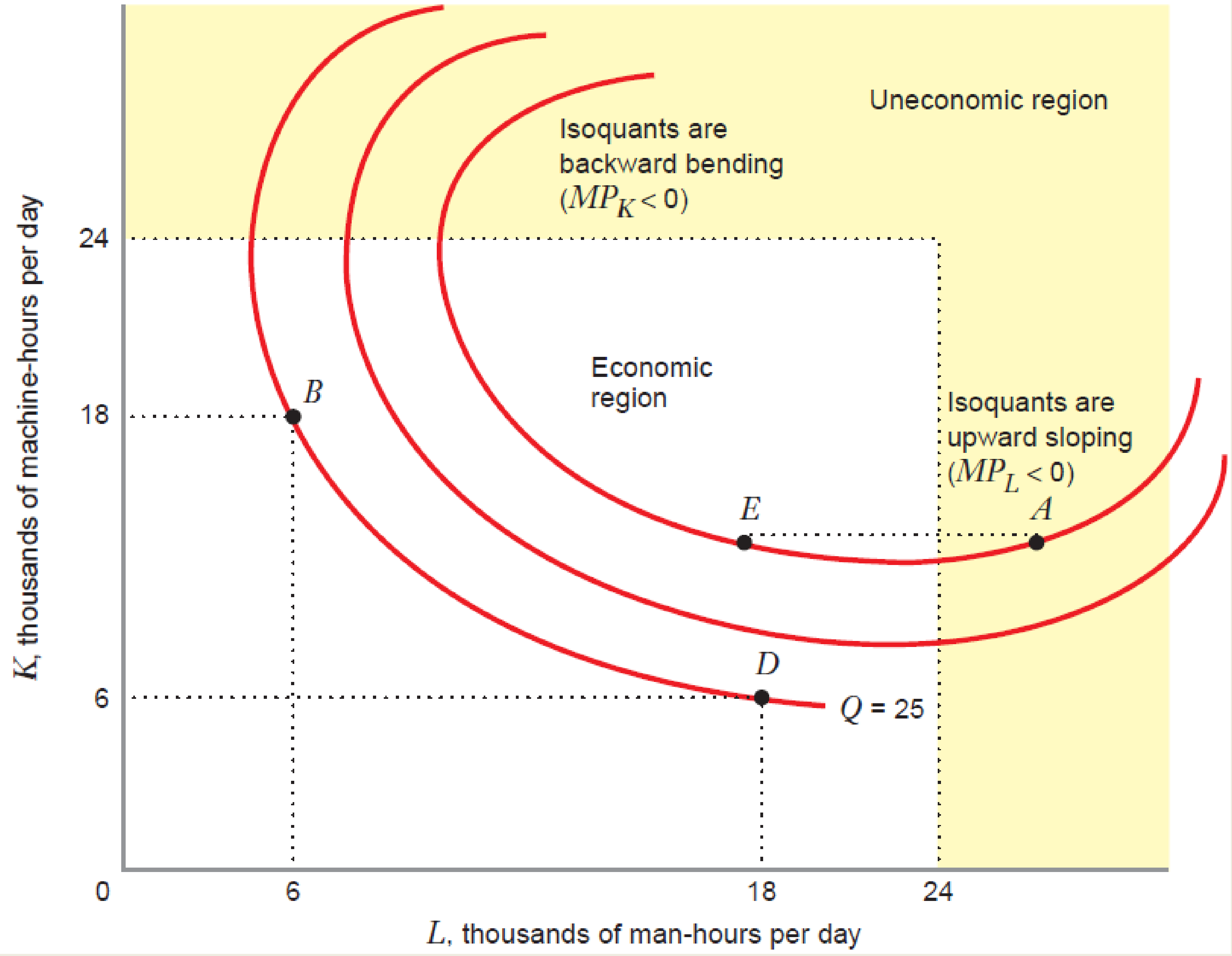
Marginal Rate of Technical Substitution

Diminishing MRTS

FIGURE 6.10
Marginal Rate of Technical Substitution of Labor for Capital ($MRTS_{L,K}$) along an Isoquant
At point *A*, the $MRTS_{L,K}$ is 2.5. Thus, the firm can hold output constant by replacing 2.5 machine-hours of capital services with an additional man-hour of labor. At point *B*, the $MRTS_{L,K}$ is 0.4. Here, the firm can hold output constant by replacing 0.4 machine-hours of capital with an additional man-hour of labor.



Marginal Rate of Technical Substitution



Elasticity of Substitution

Definition: The **elasticity of substitution**, σ, measures how the capital-labor ratio, K/L, changes relative to the change in the $MRTS_{L,K}$. **It measures of how easy is it for a firm to substitute labor for capital.**

$$\sigma = \frac{Percentage\,change\ \text{in capital - labor ratio}}{Percentage\,change\ \text{in}\ MRTS_{L,K}}$$

$$= \frac{\%\Delta\left(\frac{K}{L}\right)}{\%\Delta MRTS_{L,K}}$$

Elasticity of Substitution

Example: *Suppose that:*

- $MRTS_{L,K}^{A} = 4$, $K^{A}/L^{A} = 4$
- $MRTS_{L,K}^{B} = 1$, $K^{B}/L^{B} = 1$

$$\Delta MRTS_{L,K} = MRTS_{L,K}^{B} - MRTS_{L,K}^{A} = -3$$

$$\sigma = [\Delta(K/L)/\Delta MRTS_{L,K}]*[MRTS_{L,K}/(K/L)] = (-3/-3)(4/4) = 1$$

Elasticity of Substitution

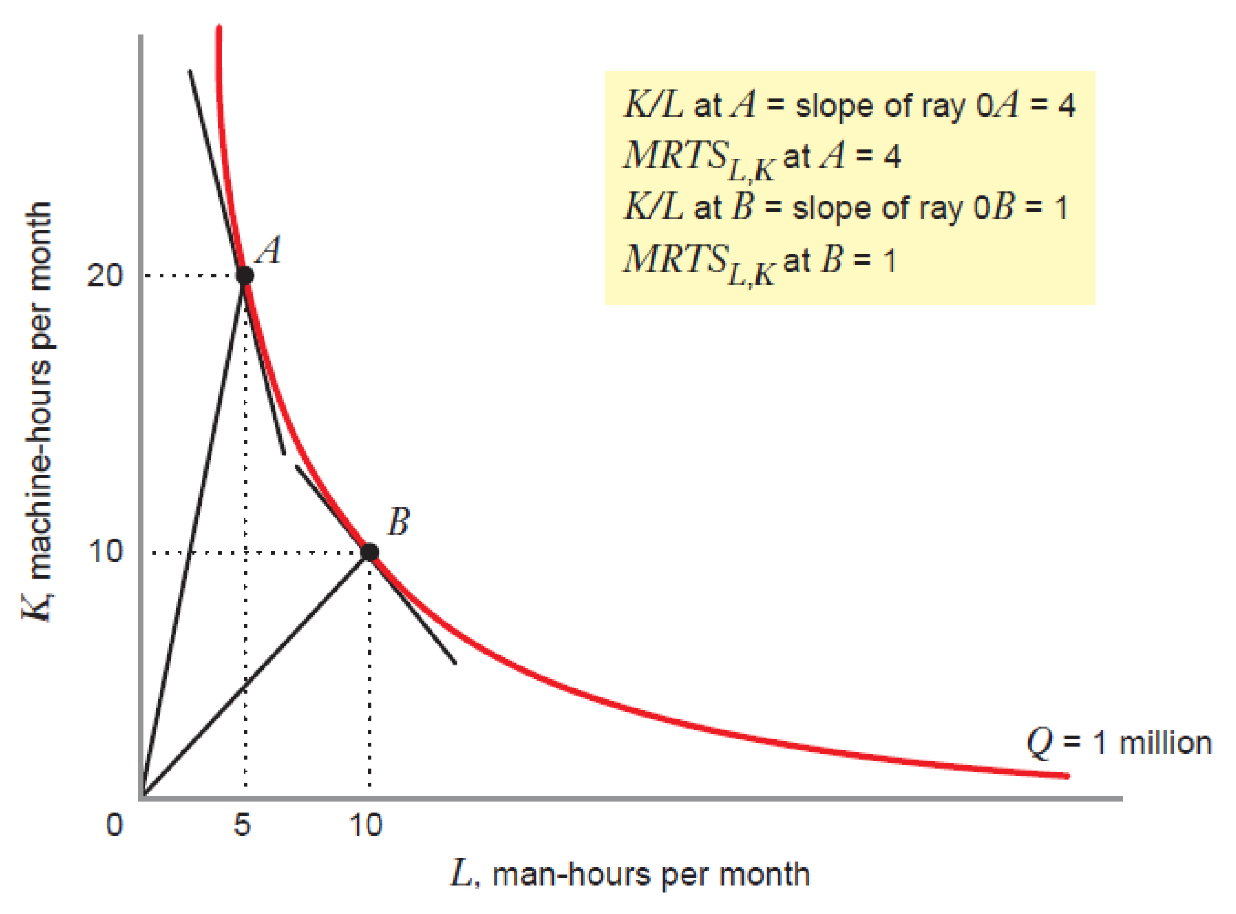
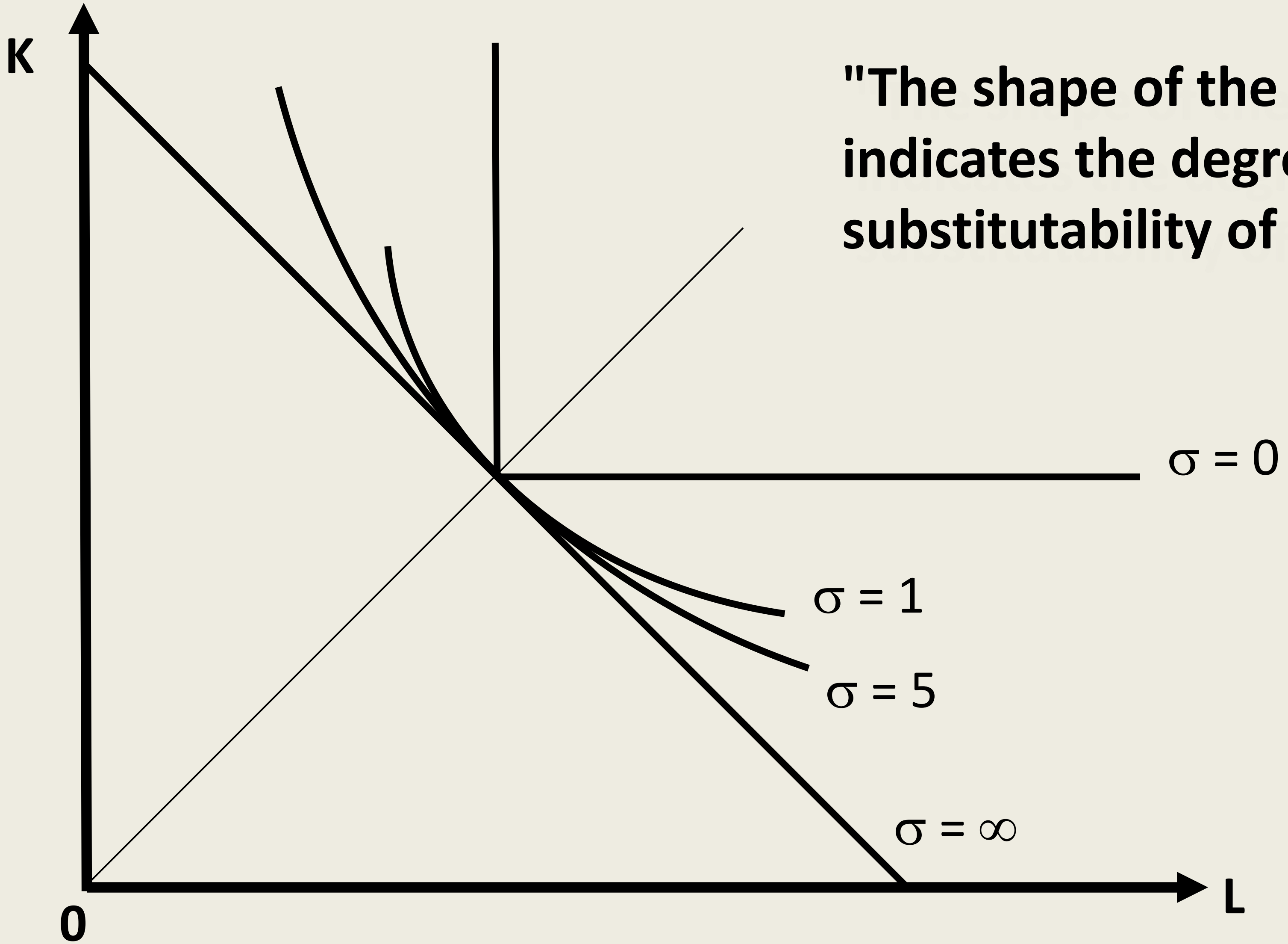


FIGURE 6.12 Elasticity of Substitution of Labor for Capital
As the firm moves from point *A* to point *B*, the capital–labor ratio *K/L* changes from 4 to 1 (–75%), as does the $MRTS_{L,K}$. Thus, the elasticity of substitution of labor for capital over the interval *A* to *B* equals 1.

Elasticity of Substitution



"The shape of the isoquant indicates the degree of substitutability of the input."

Inputs are Perfect Substitutes σ = ∞

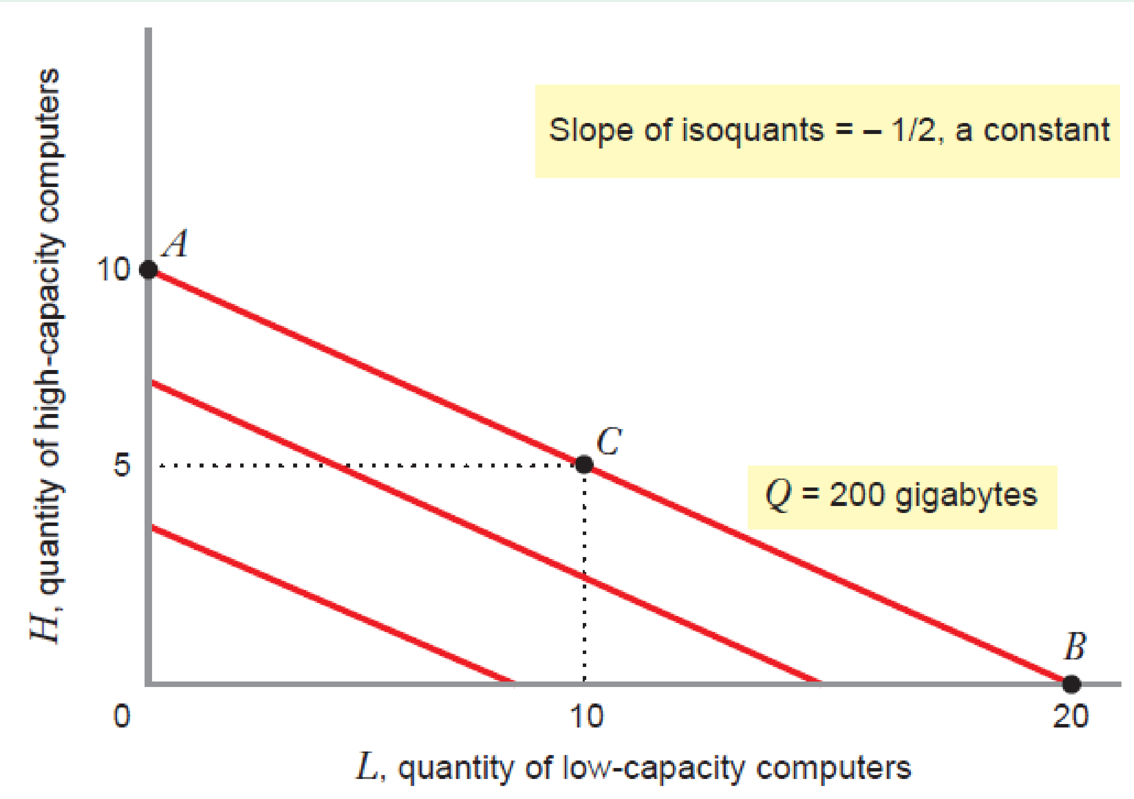


FIGURE 6.14 Isoquants for a Linear Production Function
The isoquants for a linear production function are straight lines. The $MRTS_{L,H}$ at any point on an isoquant is thus a constant.

Inputs are Perfect Complements σ = 0

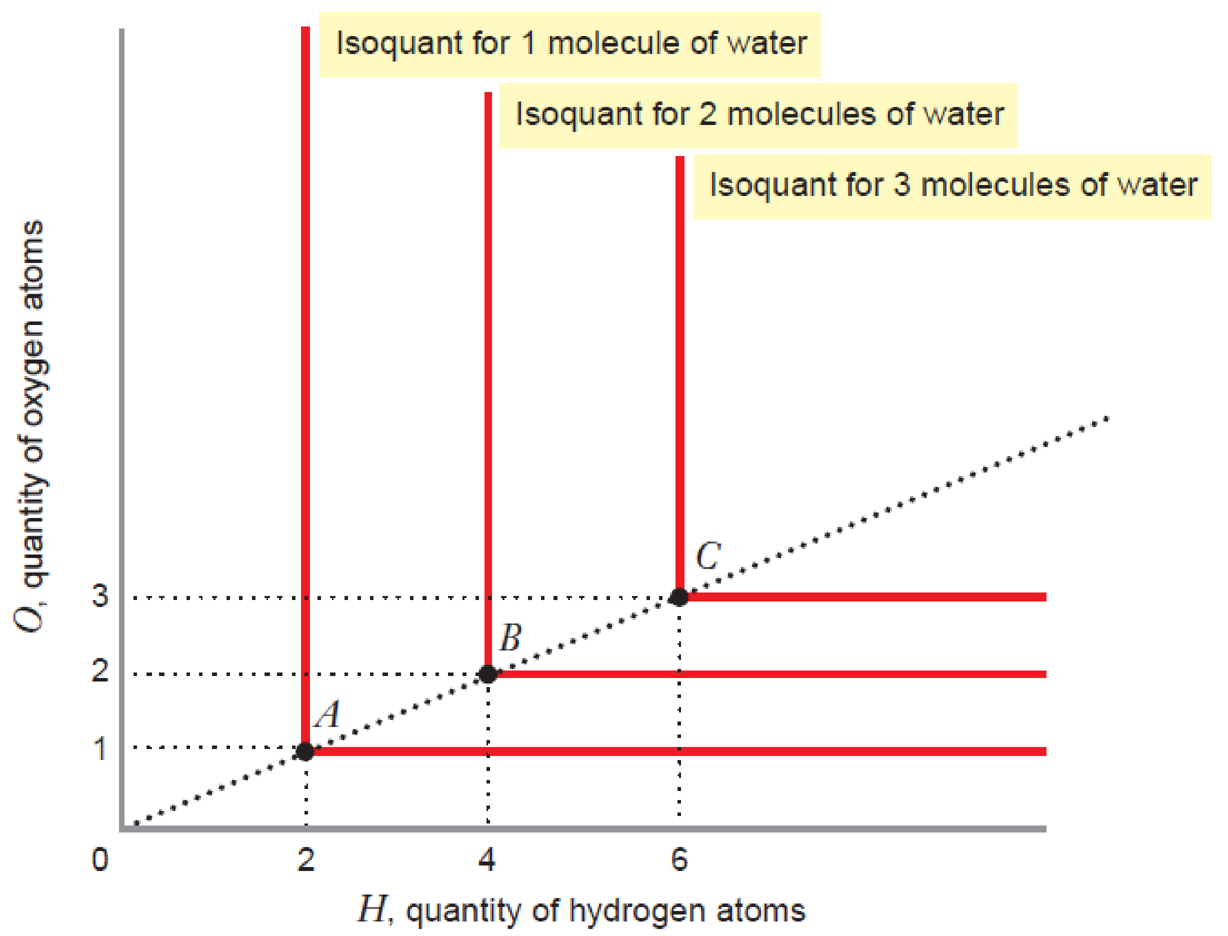


FIGURE 6.15 Isoquants for a Fixed-Proportions Production Function Two atoms of hydrogen (*H*) and one atom of oxygen (*O*) are needed to make one molecule of water. The isoquants for this production function are L-shaped, which indicates that each additional atom of oxygen produces no additional water unless two additional atoms of hydrogen are also added.

Cobb-Douglas Production Function

- $Q = AK^{\alpha}L^{\beta}$ where A, α, and β are positive.

- Cobb-Douglas function has the following properties:
  - MPK and MPL > 0.
  - MPK and MPL are diminishing w.r.t. their inputs.

  That is, $dMPK/dK < 0$ and $dMPL/dL < 0$.
  - Elasticity of Substitution = $\sigma = 1$

CES Production Function

- CES stands for Constant Elasticity of Substitution.

$$Q = \left[aL^{\frac{\sigma-1}{\sigma}} + bK^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}$$

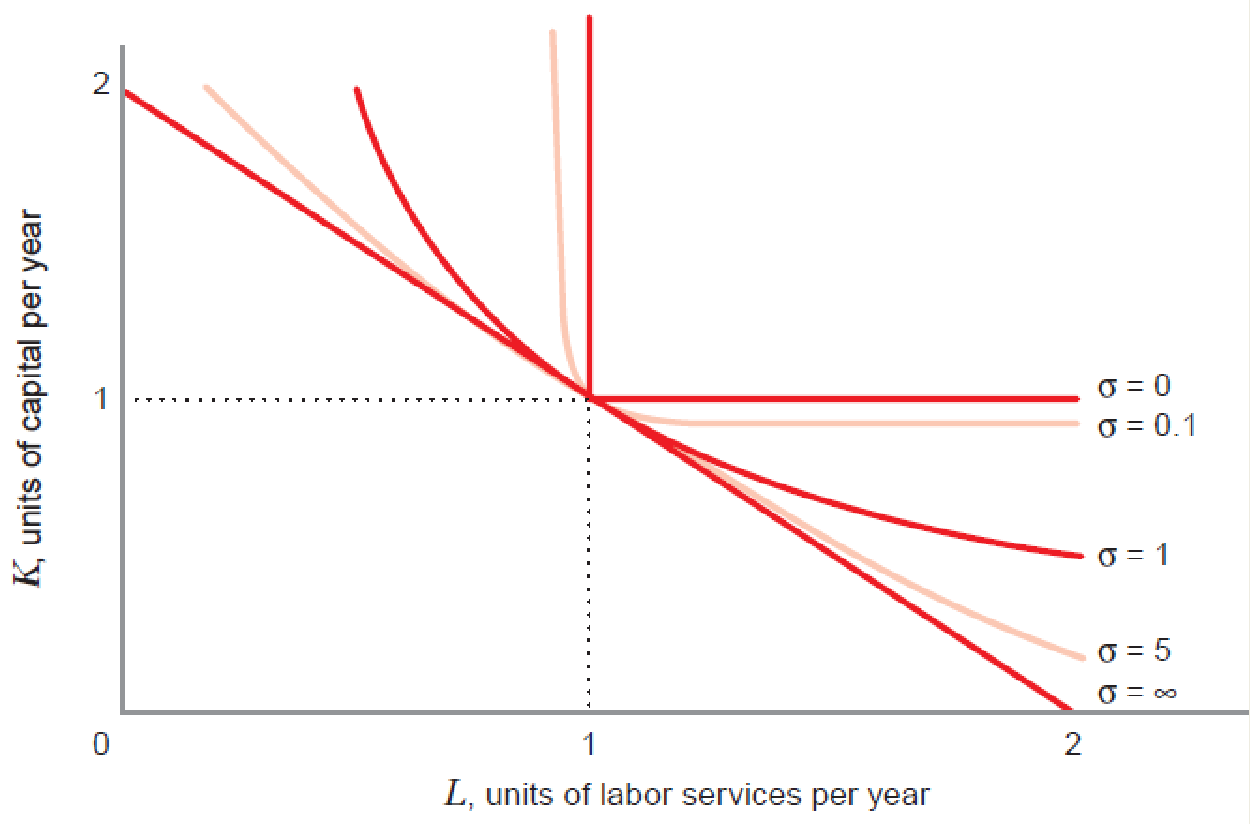
where a, b, and σ are positive.

- σ is the elasticity of substitution, σ ϵ [0, ∞).

- It combines all three types of production functions:
  - Cobb-Douglas
  - Perfect Substitutes
  - Perfect Complements

CES Production Function

FIGURE 6.17 Isoquants for the CES Production Function
This figure depicts the $Q = 1$ isoquant for five different CES production functions, each corresponding to a different value of the elasticity of substitution σ. At $\sigma = 0$, the isoquant is that of a fixed-proportions production function. At $\sigma = 1$, the isoquant is that of a Cobb–Douglas production function. At $\sigma = \infty$, the isoquant is that of a linear production function.



Returns to Scale

(Positive) **Marginal Return or Marginal Product** tells us how output will increase when one more input is added.

Returns to Scale tells us how output will increase when BOTH inputs are added.

For example, when we double inputs (K and L double), will the output double too?

Returns to Scale

Let $\lambda > 1$ represent the amount by which both inputs, labor and capital, increase.

A Production Function $F(K,L)$ is said to have

- **Increasing returns** when $F(\lambda K, \lambda L) > \lambda F(K, L)$.
- **Decreasing returns** when $F(\lambda K, \lambda L) < \lambda F(K, L)$.
- **Constant Returns** when $F(\lambda K, \lambda L) = \lambda F(K, L)$.

Returns to Scale

- **Increasing returns** when $F(\lambda K, \lambda L) > \lambda F(K, L)$:

A proportional increase in all inputs results in a greater than proportional increase in output.

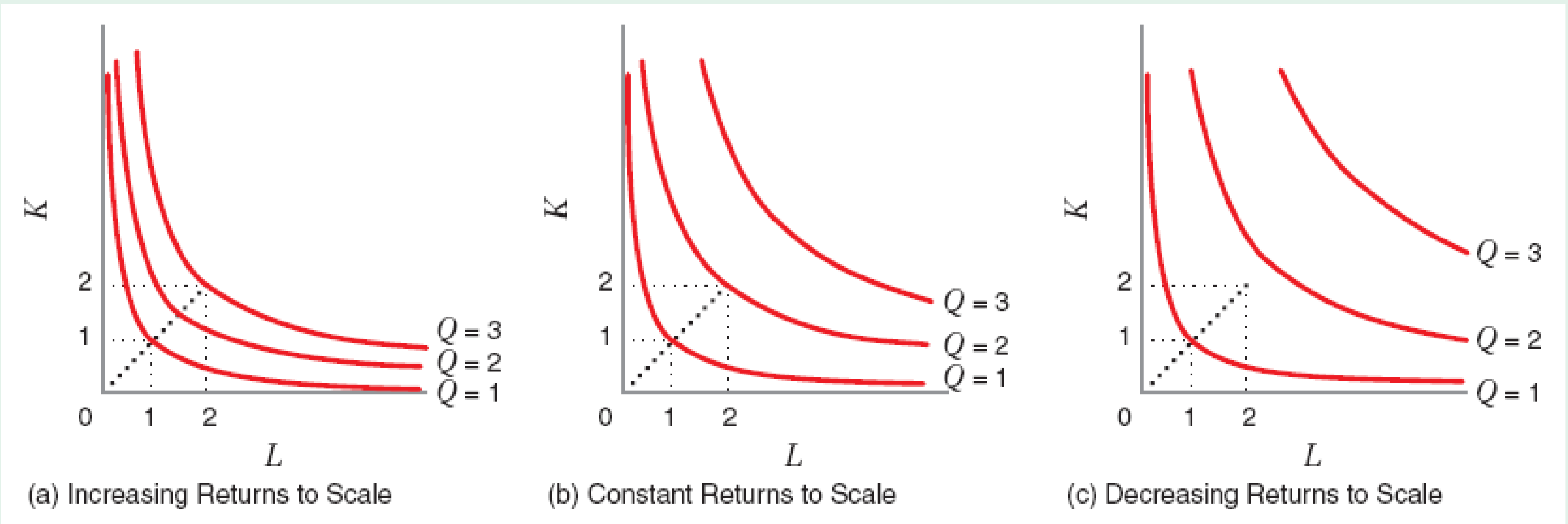
- **Decreasing returns** when $F(\lambda K, \lambda L) < \lambda F(K, L)$:

A proportional increase in all inputs results in a less than proportional increase in output.

- **Constant Returns** when $F(\lambda K, \lambda L) = \lambda F(K, L)$:

A proportional increase in all inputs results in the same proportional increase in output.

Returns to Scale



Technological Progress

Definition: **Technological progress** (or **invention**) shifts the production function by allowing the firm to achieve *more* output from a given combination of inputs (or the same output with fewer inputs).

That is, it increases "productivity", measured by MPK and MPL.

Capital-saving technological progress causes MPL to increase relative to MPK, and hence a rise in the $MRTS_{L,K}$.

Labor-saving technological progress causes MPK to increase relative to MPL, and hence a fall in the $MRTS_{L,K}$.

Neutral Technological Progress

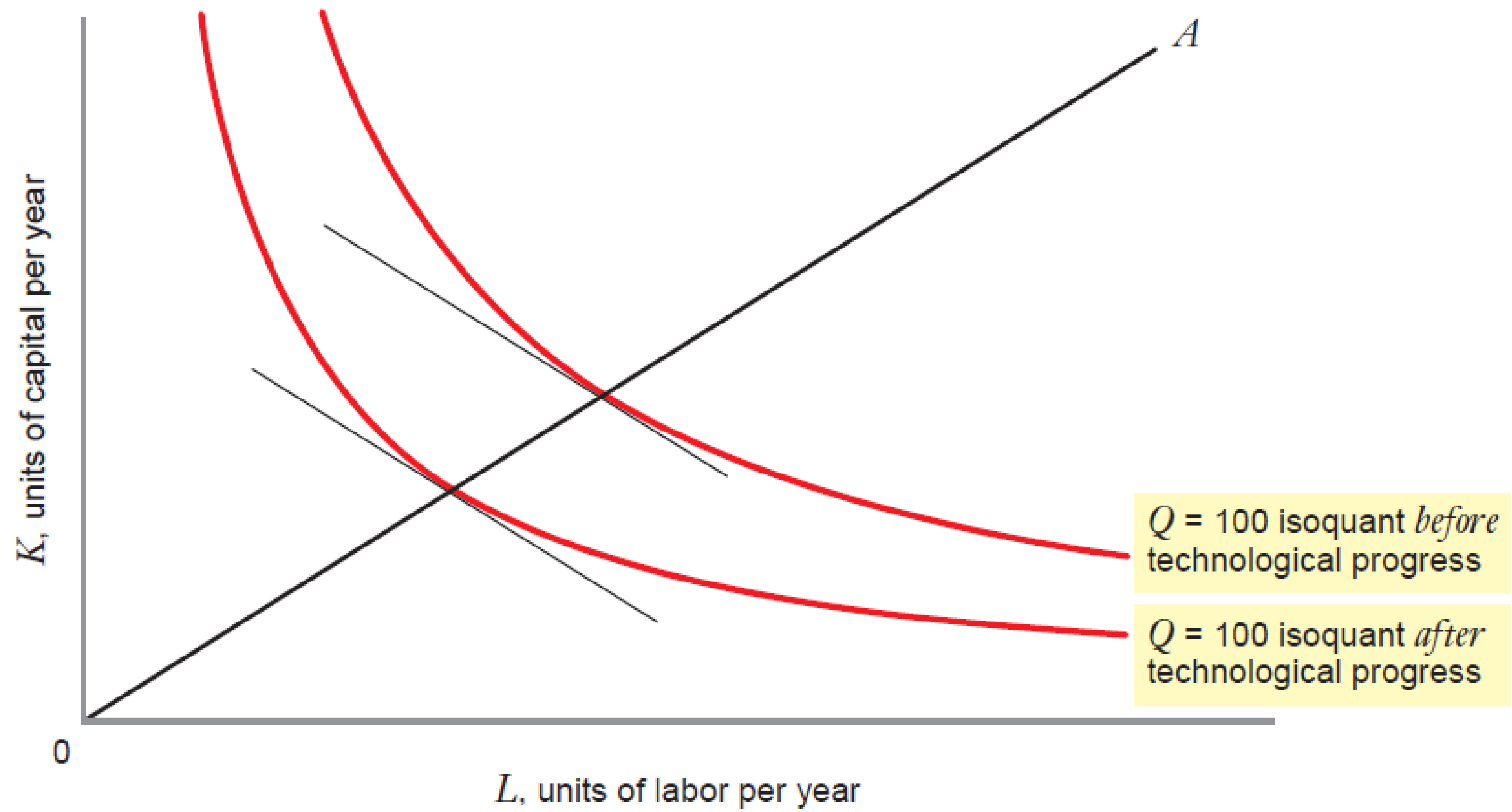
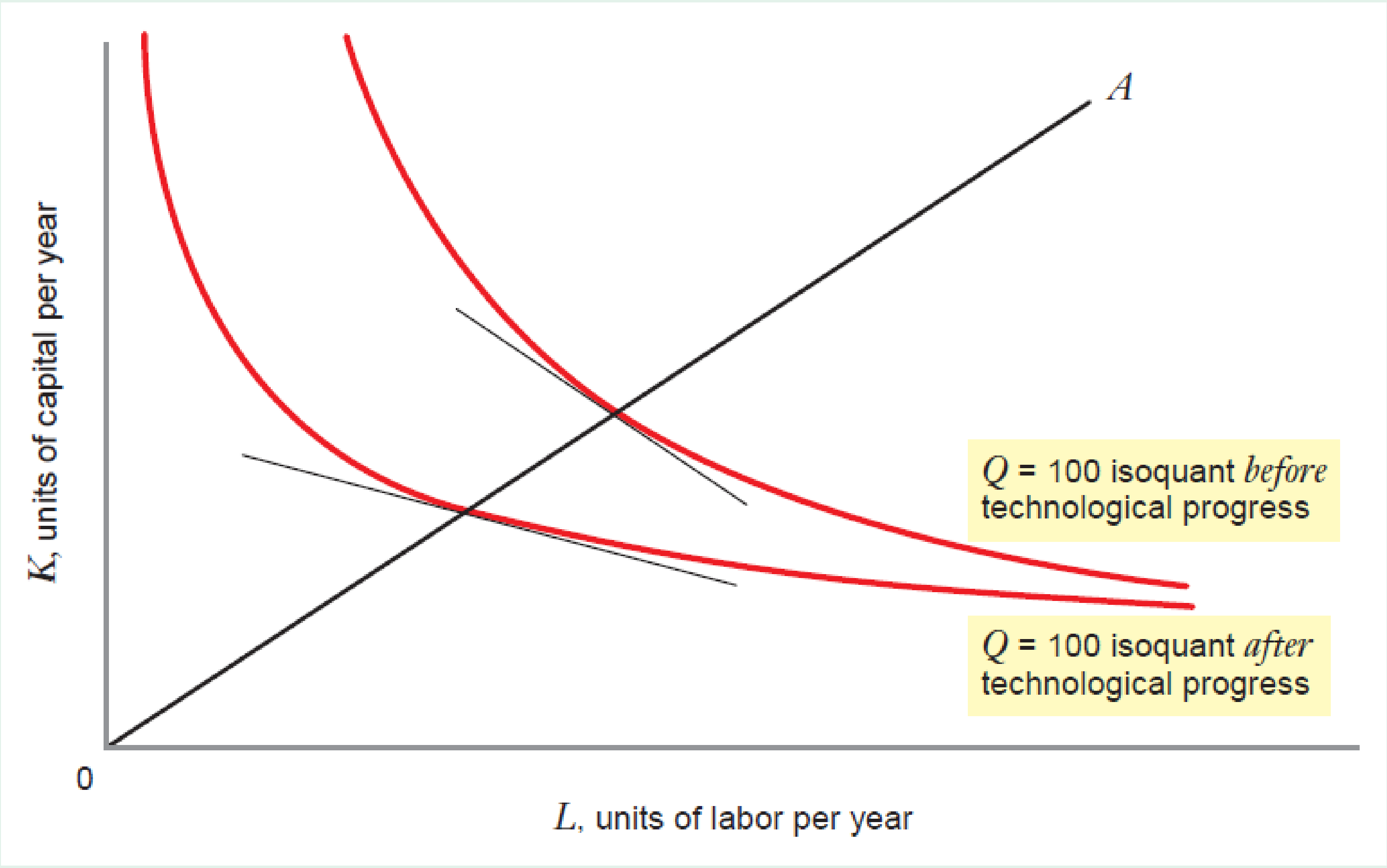


FIGURE 6.20 **Neutral Technological Progress ($MRTS_{L,K}$ Remains the Same)**
Under neutral technological progress, an isoquant corresponding to any particular level of output shifts inward, but the $MRTS_{L,K}$ (the negative of the slope of a line tangent to the isoquant) along any ray from the origin, such as 0*A*, remains the same.

Labor-Saving Technological Progress



e.g. better capital equipment

FIGURE 6.21 Labor-Saving Technological Progress (*$MRTS_{L,K}$* Decreases)
Under labor-saving technological progress, an isoquant corresponding to any particular level of output shifts inward, but the $MRTS_{L,K}$ (the negative of the slope of a line tangent to the isoquant) along any ray from the origin, such as 0*A*, goes down.

Capital-Saving Technological Progress

e.g. better skills of labor

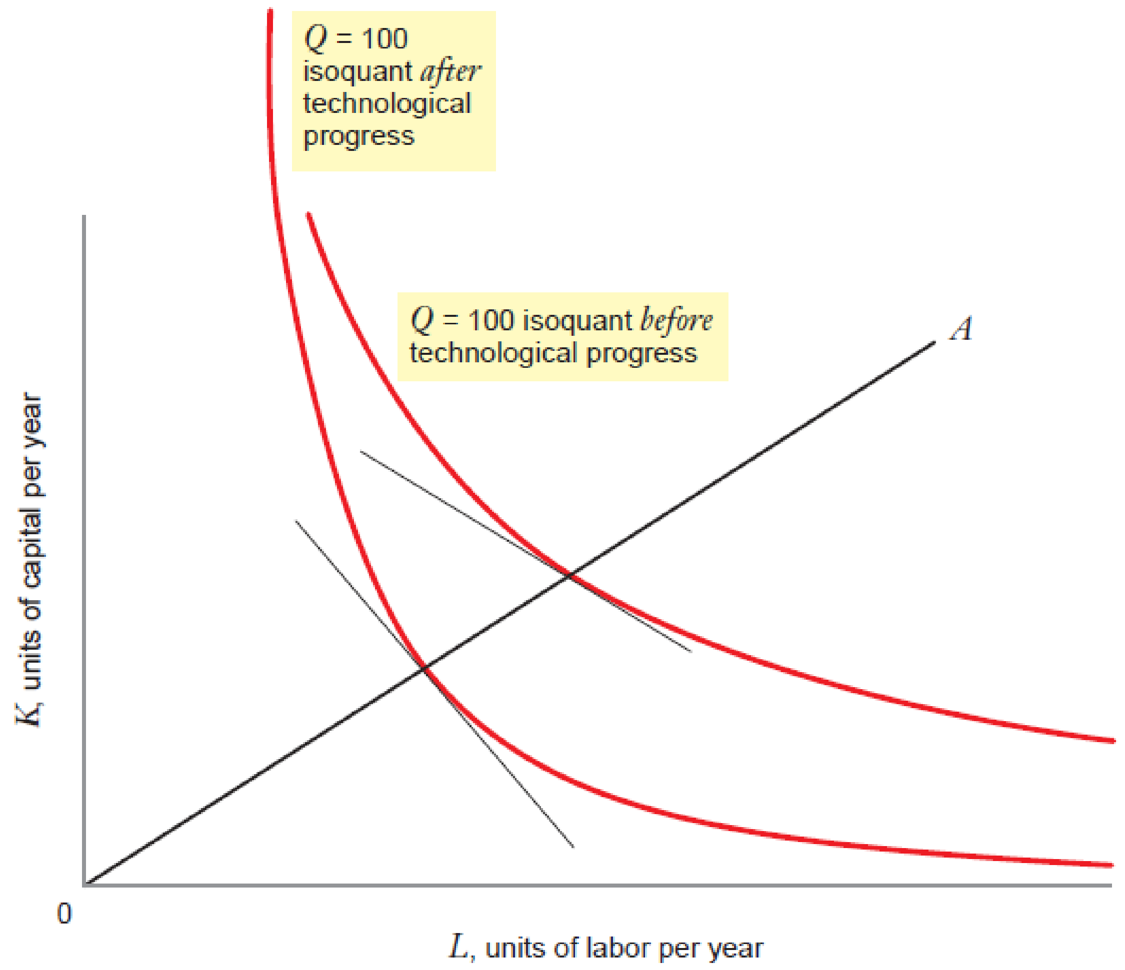


FIGURE 6.22 Capital-Saving Technological Progress ($MRTS_{L,K}$ Increases)
Under capital-saving technological progress, an isoquant corresponding to any particular level of output shifts inward, but the $MRTS_{L,K}$ (the negative of the slope of a line tangent to the isoquant) along any ray from the origin, such as 0*A*, goes up.