EE 415 / EE 418 GAME THEORY

Lecture 7 Dynamic Games of Complete Information (continued)

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2.2 Two- State Games of complete but Imperfect Information

- 2.2.A Theory: Subgame Perfection
  - As in dynamic games of complete and perfect information, we continue to assume that play proceeds in sequence of stages, with the moves in all previous stage observed before the next stage begins.
  - Unlike in the games analyzed in the previous section, however, <u>we now allow there to be simultaneous moves within each stage.</u>

  - We will analyze the following simple game, which we call a two-stage game of complete but imperfect information:

1. Player 1 and 2 simultaneously choose action a_1 and a_2 from feasible set A_1 and A_2, respectively.

2. Player 3 and 4 observe the outcome of the first stage, (a_1, a_2), and then simultaneously choose actions a_3 and a_4 from feasible sets A_3 and A_4, respectively

3. Payoff are $u_i(a_1, a_2, a_3, a_4)$ for $i = 1, 2, 3, 4$.

- Many economic problem fit this description. Three examples are bank runs, tariffs and imperfect international competition, and tournaments.
- Other economic problem can be modeled by allowing for a longer sequence of stages, either by adding players or by allowing players to move in more than stage.
- There could also be fewer players: in some applications, players 3 and 4 are players 1 and 2; in others, either player 2 or player 4 is missing.

- We solve a game from this class by using an approach in the spirit of backwards induction, but this time the first step in working backwards from the end of the game involves solving a real game (the simultaneous-move game between players 3 and 4 in stage two, given the outcome from stage one) rather than solving a single-person optimization problem as in the previous section.

- To keep things simple, in this section we will assume that for each feasible outcome of the first-stage game, (a_1, a_2), the second-stage game that remains between players 3 and 4 has a unique Nash equilibrium, denote by

$$(a_3^*(a_1, a_2), a_4^*(a_1, a_2))$$

- If players 1 and 2 anticipate that the second-stage behavior of players 3 and 4 will be given by $(a_3^*(a_1, a_2), a_4^*(a_1, a_2))$, then the first-stage interaction between players 1 and 2 amounts to the following simultaneous-move game:
  1. Players 1 and 2 simultaneously choose actions a_1 and a_2 from feasible sets A_1 and A_2, respectively.
  2. Payoffs are $u_i(a_1, a_2, a_3^*(a_1, a_2), a_4^*(a_1, a_2))$ for $i = 1, 2$.
- Suppose (a_1^*, a_2^*) is the unique Nash equilibrium of this simultaneous-move game.
- We will call $(a_1^*, a_2^*, a_3^*(a_1^*, a_2^*), a_4^*(a_1^*, a_2^*))$ the ***subgame-perfect outcome*** of this two-stage game.

- This outcome is the natural analog of the backward-induction outcome in games of complete and perfect information, and the analogy applies to both the attractive and the unattractive features of the letter.
- Players 1 and 2 should not believe a threat by players 3 and 4 that the latter will respond with actions that are not a Nash equilibrium in the remaining second-stage game, because when play actually reaches the second stage at least one of players 3 and 4 will not want to carry out such a threat.
- On the other hand, suppose that player 1 is also player 3, and that player 1 does not play a_1^* in the first stage: player 4 may then want to reconsider the assumption that player 3 (i.e.,player 1) will play $a_3^*(a_1, a_2)$ in the second stage.

2.2.B Bank Runs

- Two investors have each deposited D with a bank. The bank has invested these deposits in a long-term project.
- If the bank is forced to liquidate its investment before the project matures, a total of $2r$ can be recovered, where $D > r > D/2$.
- If the bank allows the investment to reach maturity, however, the project will pay out a total of $2R$, where $R > D$.
- There are two dates at which the investors can make withdrawals from the bank:
  - date 1 is before the bank's investment matures
  - date 2 is after

- Assume that there is no discounting.
  - If both investors make withdrawals at date 1 then each receives r and the game ends.
  - If only one investor makes a withdrawals at date 1 then that investor receives D, the other receives $2r - D$, and the game ends.
  - Finally, if neither investor makes a withdrawal at date 1 then the project matures and the investors make decisions at date 2.
  - If both investors make withdrawals at date 2 then each receives R and the game ends.
  - If only one investor makes a withdrawals at date 2 then that investor receives $2R - D$, the other receives D, and the game ends.
  - Finally, if neither investor makes a withdrawal at date 2 then the bank returns R to each investors proceed to the normal-form game at date 2.

	withdraw	don't
withdraw	r, r	$D, 2r - D$
don't	$2r - D, D$	next stage

Date 1

	withdraw	don't
withdraw	R, R	$2R - D, D$
don't	$D, 2R - D$	R, R

Date 2

- To analyze this game, we work backwards. Consider the normal-form game at date 2. Since $R > D$ (and so $2R - D > R$), "withdraw" strictly dominates "don't withdraw", so there is a unique Nash equilibrium in this game: both investors withdraw, leading to a payoff of (R,R).

- Since there is no discounting, we can simply substitute this payoff into the normal-form game at date 1, as in figure 2.2.1.

	withdraw	don't
withdraw	r, r	$D, 2r - D$
don't	$2r - D, D$	R, R

Figure 2.2.1.

- Since $r < D$ (and so $2r - D < r$), this one-period version of the two-period game has two pure-strategy Nash equilibria:
  1. both investors withdraw, leading to a payoff of (r,r)
  2. both investors do not withdraw, leading to a payoff of (R,R).

Thus, the original two-period bank-runs game has two subgame-perfect outcome:

1. both investors withdraw at date 1, yielding payoffs of (r,r)
2. both investors do not withdraw at date 1 but do withdraw at date 2, yielding payoffs of (R,R) at date 2.

The first of these outcomes can be interpreted as a run on the bank.

- If investor 1 believes that investor 2 will withdraw at date 1 that investor 1's best response is to withdraw as well, even though both investors would be better off if they waited until date 2 to withdraw.

2.2.C TARIFFS AND IMPERFECT INTERNATIONAL COMPETITION

- Consider two identical countries, denoted by $i = 1, 2$.
- Each country has a government that chooses a tariff rate, a firm that produces output for both home consumption and export, and consumers who buy on the home market from either the home firm or the foreign firm.
- If the total quantity on the market in country i is Q_i, then the market-clearing price is $P_i(Q_i) = a - Q_i$.
- The firm in country i produces h_i for home consumption and e_i for export.
- Thus, $Q_i = h_i + e_j$.
- The firms have a constant marginal cost, c and no fixed costs.

- Thus, the total cost of production for firm i is

$$C_i(h_i, e_i) = c(h_i + e_i)$$

- The firms also incur tariff costs on exports:
  - if firm i exports e_i to country j when government j has set the tariff rate t_j, then firm i must pay $t_j e_j$ to government j.
- The timing of the game is as follows.
  - First, the governments simultaneously choose tariff rate, t_1 and t_2.
  - Second, the firms observe the tariff rates and simultaneously choose quantities for home consumption and for export, (h_1, e_1) and (h_2, e_2).
  - Third, payoffs are profit to firm i and total welfare to government i. *The total welfare to country i is the sum of consumer surplus from consumers in i, the profit earned by firm i, and tariff revenue*

$$\pi_i(t_i, t_j, h_i, e_i, h_j, e_j) = [a-(h_i+e_j)]h_i + [a-(e_i+h_j)]e_i \\ -c(h_i+e_i) - t_j e_i,$$

$$W_i(t_i, t_j, h_i, e_i, h_j, e_j) = \frac{1}{2}Q_i^2 + \pi_i(t_i, t_j, h_i, e_i, h_j, e_j) + t_i e_j.$$

- Suppose the governments have chosen the tariff t_1 and t_2. If $(h_1^*, e_1^*, h_2^*, e_2^*)$ is Nash equilibrium in the remaining (two-market) game between firms 1 and 2 then, for each *i*,

  (h_i^*, e_i^*) must solve

$$\max_{h_i, e_i \geq 0} \pi_i(t_i, t_j, h_i, e_i, h_j^*, e_j^*).$$

- Since $\pi_i(t_i, t_j, h_i, e_i, h_j^*, e_j^*)$ can be written as the sum of firm *i*'s profits on market *i* and firm *i*'s profits on market *j*, firm *i*'s two-market optimization problem simplifies into a pair of problems, one for each market: h_i^* must solve

$$\max_{h_i \geq 0} h_i[a - (h_i + e_j^*) - c],$$

and e_i^* must solve

$$\max_{e_i \geq 0} e_i[a - (e_i + h_j^*) - c] - t_j e_i$$

Assuming $e_j^* \le a-c$, we have

$$h_i^* = \frac{1}{2}(a - e_j^* - c), \qquad (2.2.1)$$

and assuming $h_j^* \le a-c-t_j$, we have

$$e_i^* = \frac{1}{2}(a - h_j^* - c - t_j). \qquad (2.2.2)$$

- Both of the best-response functions (2.2.1) and (2.2.2) must hold for each $i = 1,2$.
- Thus, we have four equations in the four unknowns $(h_1^*, e_1^*, h_2^*, e_2^*)$
- These equations simplify into two sets of two equations in two unknowns. The solutions are

$$h_i^* = \frac{a-c+t_i}{3} \quad \text{and} \quad e_i^* = \frac{a-c-2t_j}{3} \qquad (2.2.3)$$

- In the equilibrium described by (2.2.3), the governments' tariff choices make marginal costs asymmetric. (contrast to symmetric marginal cost Cournot model)
- On market *i*, for instance, firm *i*'s marginal cost is c but firm *j*'s is $c+t_i$. Since firm *j*'s cost is higher it wants to produce less. But if firm *j* is going to produce less, then market-clearing price will be higher, so firm *i* wants to produce more, in which case firm *j* wants to produce even less.
- Thus in equilibrium, h_i^* increases in t_i and e_j^* decreases in t_i, as in (2.2.3)

Having solved the second-stage game, we now look at the first stage game (interaction b/w gov) ;

- First, the governments simultaneously choose tariff rate t_1 and t_2.
- Second, payoffs are $W_i(t_i, t_j, h_1^*, e_1^*, h_2^*, e_2^*)$ for government $i=1,2$, where h_i^* and e_i^* are function of t_1 and t_2 described in (2.2.3).

- To simplify the notation, we will suppress the dependence of h_i^* on t_i and e_i^* on t_j :

  let $W_i^*(t_i, t_j)$ denote $W_i(t_i, t_j, h_1^*, e_1^*, h_2^*, e_2^*)$, the payoff to government i when it chooses the tariff rate t_i government j chooses t_j , and firm i and j play Nash equilibrium given in (2.2.3).

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- If (t_1^*, t_2^*) is Nash equilibrium of this game between governments then, for each i, t_i^* must solve

$$\max_{t_i \geq 0} W_i^*(t_i, t_j^*)$$

But $W_i^*(t_i, t_j^*)$ equals

$$\frac{(2(a-c)-t_i)^2}{18} + \frac{(a-c+t_i)^2}{9} + \frac{(a-c-2t_j^*)^2}{9} + \frac{t_i(a-c-2t_i)}{3}$$

so

$$t_i^* = \frac{a-c}{3}$$

for each *i*, independent of t_j^*.

- Thus in this model , choosing a tariff rate of *(a-c)/3* is a dominant strategy for each government.

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Substituting $t_i^* = t_j^* = (a-c)/3$ into (2.2.3) yields

$$h_i^* = \frac{4(a-c)}{9} \quad \text{and} \quad e_i^* = \frac{a-c}{9}$$

as the firms' quantity choices in the second stage.

- Thus, the subgame-perfect outcome of this tariff game is

$$t_1^* = t_2^* = (a-c)/3,$$
$$h_1^* = h_2^* = 4(a-c)/9,$$
$$e_1^* = e_2^* = (a-c)/9.$$

- In the subgame-perfect outcome the aggregate quantity on each market is $5(a-c)/9$. If the governments had chosen tariff rates equal to zero, however, then aggregate quantity on each market would have been $2(a-c)/3$.

- Thus, the consumer s' surplus on market *i* is lower when the governments choose their dominant strategy tariffs than it would be if they chose zero tariffs.

- In fact, zero tariffs are socially optimal, in the sense that $t_1 = t_2 = 0$ is the solution to

$$\max_{t_1, t_2 \geq 0} W_1^*(t_1, t_2) + W_2^*(t_2, t_1)$$

so there is an incentive for the governments to sign a treaty in which they commit to zero tariffs.