Chapter 3 Application of Differentiation

Section 3.1 Related Rates

In a related rates problem the idea is the compute the rate of change of one quantity in terms of rate of change of another quantity (which may be more easily measured). The procedure is to find an equation that relates the two quantities and then use the Chain Rule to differentiate both sides with respect to time.

Example 1: Consider a triangle whose height is shrinking at a rate of 1 cm/sec and whose base is growing at 2 cm/sec. How fast is the area changing when both height and base are 4 cm?

Example 2: A spotlight on the ground shines on a wall 12 m away. If a man 2 m tall walks from the spotlight toward the building at a speed of 1.6 m/s, how fast is the length of his shadow on the building decreasing when he is 4 m from the building?

Section 3.2 Indeterminate Forms and L'Hospital's Rule

In general, if we have a limit of the form $\lim_{x\to a}\frac{f(x)}{g(x)}$ where both $f(x)\to 0$ and $g(x)\to 0$ as $x\to a$, then this limit may or may not exist and is called an ***indeterminate form of type 0/0.*** In this section we introduce a systematic method, known as L'Hospital's Rule, for the evaluation of indeterminate forms.

Moreover, if we have a limit of the form $\lim_{x\to a}\frac{f(x)}{g(x)}$ where both $f(x)\to\infty$ (or $-\infty$) and $g(x)\to\infty$ (or $-\infty$) as $x\to a$, then the limit may or may not exist and is called an ***indeterminate form of type*** ∞/∞**.** L'Hospital's Rule also applies to this type of indeterminate form.

L'Hospital's Rule: Suppose f and g are differentiable and $g'(x)\neq 0$ near a (except possibly at a). Suppose that

$$\lim_{x\to a} f(x)=0 \quad \text{and} \quad \lim_{x\to a} g(x)=0$$

or that

$$\lim_{x\to a} f(x)=\pm\infty \quad \text{and} \quad \lim_{x\to a} g(x)=\pm\infty$$

(In other words, we have an indeterminate form of type 0/0 or ∞/∞.) Then

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$$

if the limit on the right side exists (or is ∞ or $-\infty$).

Example 1: Find $\lim_{x\to 1}\frac{1-x}{\ln x}$.

Example 2: Find $\lim_{x\to\infty}\frac{e^x}{x^2}$.

Example 3: Find $\lim_{x\to 0}\frac{\tan x - x}{x^3}$.

Example 4: Find $\lim_{x\to 0}\frac{\sin x}{1-\cos x}$.

Indeterminate Products

If $\lim_{x\to a} f(x)=0$ and $\lim_{x\to a} g(x)=\infty$ (or $-\infty$), then it isn't clear what the value of $\lim_{x\to a} f(x)g(x)$, if any, will be. This kind of limit is called an *indeterminate form of type* $0\cdot\infty$. We can deal with it by writing the product *fg* as a quotient

$$fg=\frac{f}{1/g} \quad \text{or} \quad fg=\frac{g}{1/f}.$$

This converts the given limit into an indeterminate form of type 0/0 or ∞/∞ so that we can use L'Hospital's Rule.

<u>Example 5</u>: Evaluate $\lim_{x\to 0^+} x\ln x$

<u>Example 6</u>: Evaluate $\lim_{x\to\infty} x\tan(1/x)$

Indeterminate Difference

If $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = \infty$, then

$$\lim_{x\to a}\left[f(x) - g(x)\right]$$

is called an *indeterminate form of type* $\infty - \infty$. In this case, we try to convert the difference into a quotient (for instance, by using a common denominator, or rationalization, or factoring out a common factor) so that we have an indeterminate form of type 0/0 or ∞/ ∞.

Example 7: Compute $\lim_{x\to(\pi/2)^-}(\sec x - \tan x)$.

Example 8: Evaluate $\lim_{x\to 1}\left(\frac{1}{\ln x} - \frac{1}{x-1}\right)$.

Indeterminate Power

Several indeterminate forms arise from the limit

$$\lim_{x\to a}\left[f(x)\right]^{g(x)}$$

1. $\lim_{x\to a} f(x)=0$ and $\lim_{x\to a} g(x)=0$ type 0^0
2. $\lim_{x\to a} f(x)=\infty$ and $\lim_{x\to a} g(x)=0$ type ∞^0
3. $\lim_{x\to a} f(x)=1$ and $\lim_{x\to a} g(x)=\pm\infty$ type 1^∞

Each of these three cases can be treated either by taking the natural logarithm:

$$\text{let } y=\left[f(x)\right]^{g(x)}\text{ , then } \ln y = g(x)\ln\left(f(x)\right)$$

or by writing the function as an exponential:

$$\left[f(x)\right]^{g(x)} = e^{g(x)\ln f(x)}$$

In either method we are led to the indeterminate product $g(x)\ln f(x)$, which is of type $0\cdot\infty$.

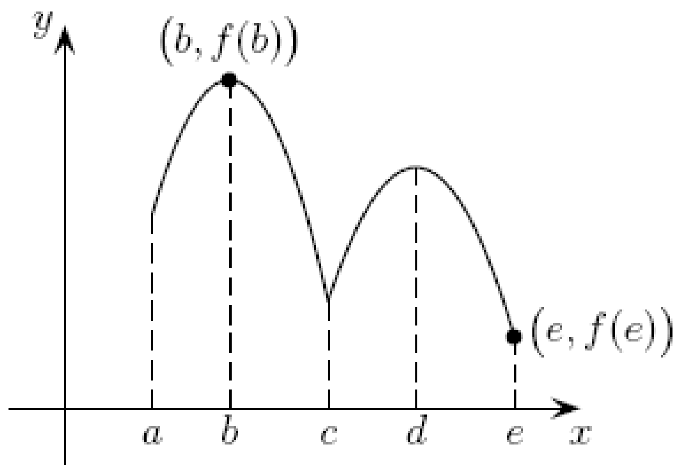
<u>Example 10</u>: Find $\lim_{x\to 0^+} x^x$.

Example 11: Find $\lim_{x\to\infty}(1-2x)^{1/x}$.

Section 3.3: Maximum and Minimum Values of a Function

<u>Definition</u>: A function f has an **absolute maximum** (or global maximum) at c if $f(c) \geq f(x)$ for all x in D, where D is the domain of f. The number $f(c)$ is called the **maximum value** of f on D.

Similarly, f has an **absolute minimum** at c if $f(c) \leq f(x)$ for all x in D and the number $f(c)$ is called the **minimum value** of f on D. The maximum and minimum values of f are called the **extreme value** of f.

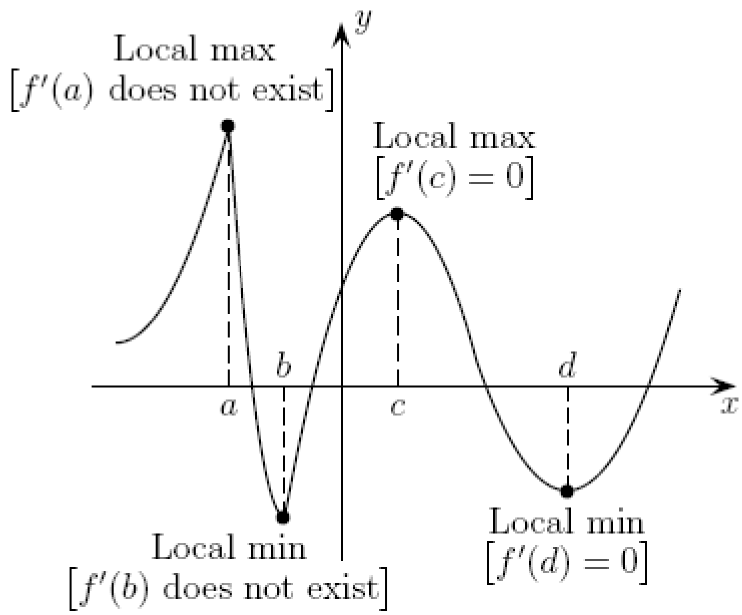


From the figure, f has an absolute maximum (or global maximum) at b and absolute minimum (or global minimum) at e.

<u>Definition</u>: A function f has a **local maximum** (or relative maximum) at c if $f(c) \geq f(x)$ when x is near c. [This means that $f(c) \geq f(x)$ for all x in some *open interval* containing c.] Similarly, f has a **local minimum** at c if $f(c) \leq f(x)$ when x is near c.

If we consider the interval (a,c), f has a local maximum at b as b is the largest value in this interval and f (b) is called a local maximum value.

In the interval (b,d), f has a local minimum at c as c is the smallest value in this interval and f (c) is called a local maximum value.



The figure above illustrates that a local extreme value can occur at a point in the domain of function at which either the graph of the function has a horizontal tangent line or function is not differentiable.

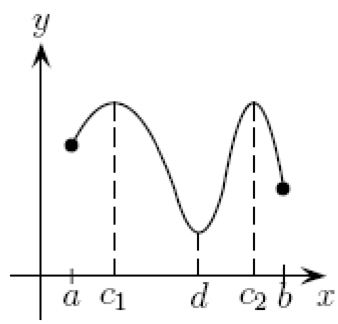
Fermat's Theorem:

If f has a local maximum or minimum at c, and if $f'(c)$ exists, then $f'(c)=0$.

Note: The reverse is NOT TRUE !

Example 1: Where does the function have an absolute extrema?

(1)



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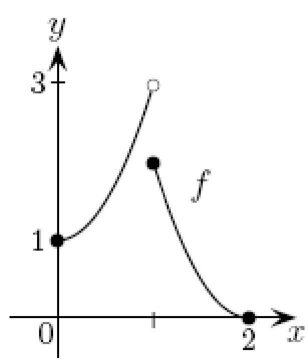

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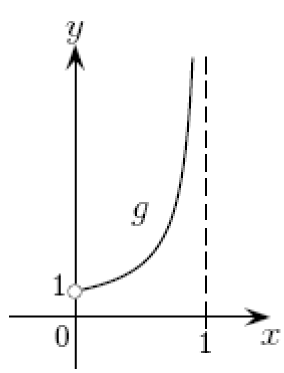


Definition : A critical number (or critical value) of a function *f* is a number *c* in the domain of *f* such that either $f'(c)=0$ 0 or $f'(c)$ does not exist.

Example 2: Find the critical numbers of the function.

(1) $f(x)=(6x-1)^{2/3}$

(2) $f(x)=3x^2+2x-1$

(3) $f(x)=2x^3-3x^2-12x+5$

(4) $f(x)=(x^2-1)^{-1}$

Recall: Fermat's Theorem:

If f has a local maximum or minimum at c, and if $f'(c)$ exists, then $f'(c)=0$.

Note: The Fermat's Theorem can be rephrased as follows.

If f has a local maximum or minimum at c , then c is a critical number of f .

The Closed Interval Method:

To find the absolute maximum and minimum values of a continuous function f on a closed interval $[a,b]$:

(1) Find the value of f at the critical numbers of f in (a, b).

(2) Find the value of f at the endpoints of the interval.

(3) The largest of the value from step 1 and 2 is the *absolute maximum value*.

The smallest of the value from step 1 and 2 is the *absolute minimum value*.

Example 4: Find the absolute maximum and minimum value of the function.

(1) $f(x)=x^3-8x+1 \quad ; -3\le x\le 3$

(2) $f(x)=\left(x^2-1\right)^3 \quad ; x\in[-1,2]$

Section 3.4: Intervals of Increase and Decrease; Concavity

Section 3.5: Relative Extrema; First and Second Derivative Tests

Increasing / Decreasing Test:

(a) If $f'(x) > 0$ on an interval, then f is increasing on that interval.

(b) If $f'(x) < 0$ on an interval, then f is decreasing on that interval.

Example 1: Find where the function *f* is increasing and where it is decreasing.

$(a)\ f(x) = x^3 - 12x + 1$

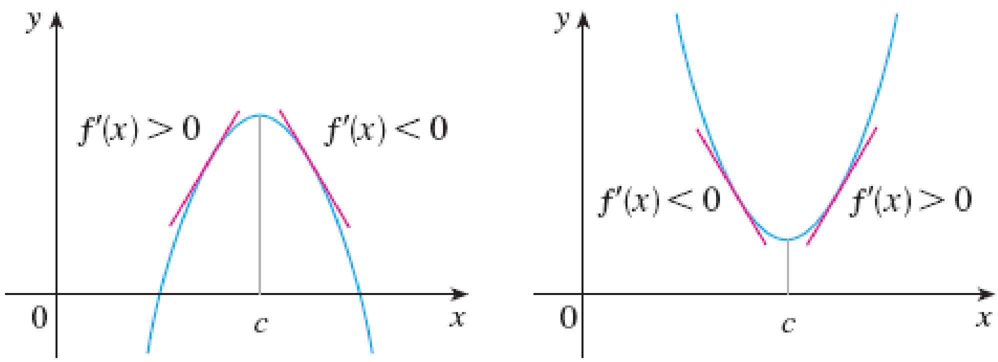
$(b)\ f(x) = \dfrac{x^4}{4} + x^3 - 2x^2$

<u>The First Derivative Test</u>: Suppose that c is a critical number of a continuous function f.

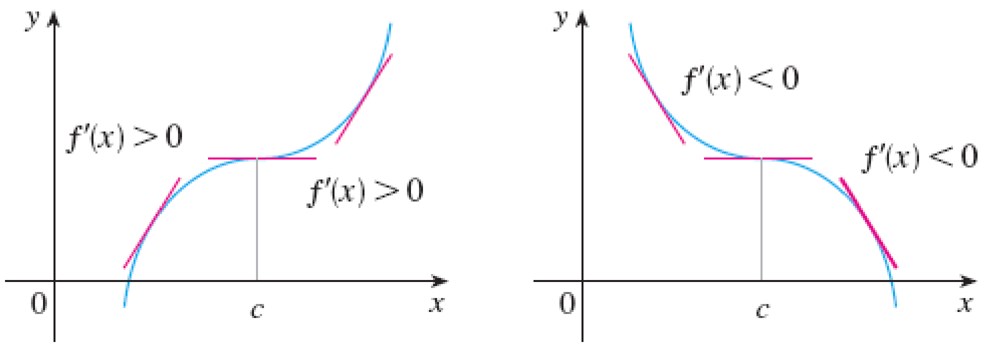
(a) If f' changes from positive to negative, then f has a local maximum at c.

(b) If f' changes from negative to positive, then f has a local minimum at c.

(c) If f' does not change sign at c, then f has no local max or min at c.



(a) Local maximum (b) Local minimum



(c) No maximum or minimum (d) No maximum or minimum

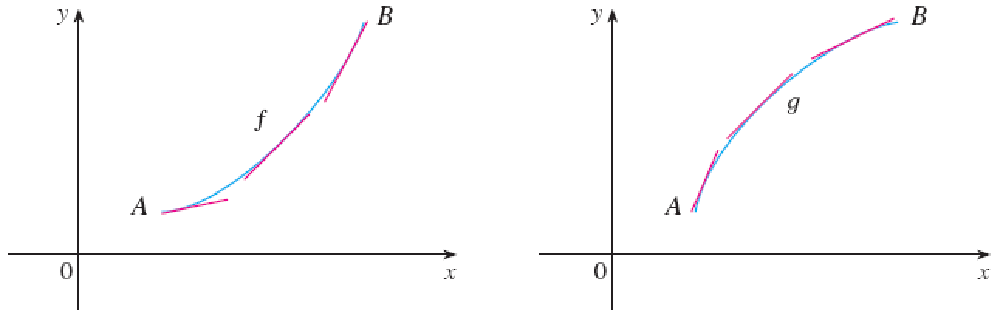
Example 2: Find the local (or relative) maximum and minimum.

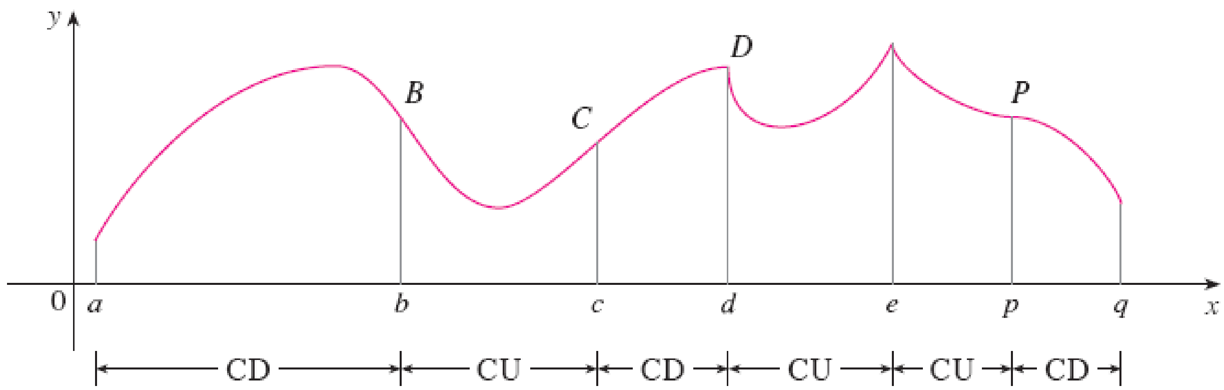
$(a)\ f(x) = x^3 - 12x + 1$

$(b)\ f(x) = \frac{x^4}{4} + x^3 - 2x^2$

What does f'' say about f'?

Definition: If the graph of f lies above all of its tangents on an interval I, then it is called *concave upward* on I. If the graph of f lies below all of its tangents on I, it is called *concave downward* on I.





Concavity Test:

(a) If $f''(x) > 0$ for all x in I, then the graph of f is concave upward.

(b) If $f''(x) < 0$ for all x in I, then the graph of f is concave downward.

Definition: A point P on a curve $y = f(x)$ is called an *inflection point* if f is continuous there and the curve changes from concave upward to concave downward or from concave downward to concave upward at P.

Example 3: Find the intervals of concavity and the inflection points of $f(x) = x^3 - 12x + 1$

<u>The Second Derivative Test</u>: Suppose f'' is continuous near *c*.

(a) If $f'(c)=0$ and $f''(c)>0$, then f has a local minimum at *c*.

(b) If $f'(c)=0$ and $f''(c)<0$, then f has a local maximum at *c*.

<u>Example 4</u>: Find the local maximum and minimum of the function $f(x)=x^3-12x+1$

Example 5:

- Find the interval of increase and decrease.
- Find the local maximum and minimum values.
- Find the intervals of concavity and the inflection points.
- Use the above information to sketch the graph.

$(a)\ f(x) = 2 + 3x - x^3$

$(b)\ f(x) = x^3 + 6x^2 + 9x$

$(c)\ f(x) = 3x^{2/3} - x$

Section 3.6: Applied Maximum and Minimum Problems (Optimization Problems)

Here we try to optimize (i.e. max or min) a quantity given a set of restraints (eg. Minimize cost, maximize profit).

Each problem is solved in two parts.

Step 1: Find the function of the quantity to be optimized, this usually involves fining objective equations and constraint equations.

Note: The resulting function can only have one variable.

Step 2: Find the global extreme points of that function (i.e. find the absolute maximum and minimum).

Recall:

The Closed Interval Method: To find the absolute maximum and minimum values of a continuous function f on a closed interval $[a,b]$, we use the closed interval method:

(1) Find the value of f at the critical numbers of f in (a, b).

(2) Find the value of f at the endpoints of the interval.

(3) The largest of the value from step 1 and 2 is the *absolute maximum value*.

The smallest of the value from step 1 and 2 is the *absolute minimum value*.

Example 1: Find two positive numbers that have a product of 10 and the sum is as small as possible.

Example 2: Farmer Bob has 300 m of fencing. He wants to build a rectangular pig-pen. If he uses the barn wall as one side of the pen, what dimensions give the largest area for the pigs?

Example 3: A rectangular box with a square bottom is made from these different materials:

Bottom : \$3/m^2

Sides : \$2/ m^2

Top : \$1/ m^2

Design a box holding 216 m^3 of material that minimize the cost.

Section 3.7: Rolle's Theorem; Mean Value Theorem

Rolle's Theorem:

Rolle's Theorem: Let f be a function that satisfies the following three hypotheses:

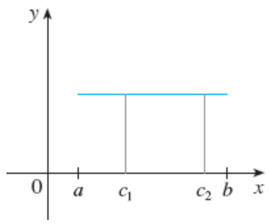
(1) f is continuous on the closed interval $[a,b]$.

(2) f is differentiable on the open interval (a, b).

(3) $f(a) = f(b)$.

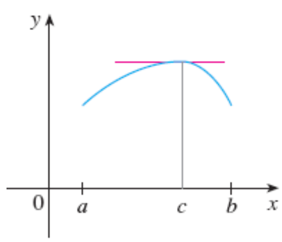
Then there is a number c in (a, b) such that $f'(c) = 0$.

CASE I: $f(x) = k$ where k is a constant



(a)

CASE II: $f(x) > f(a)$ for some x in (a,b)



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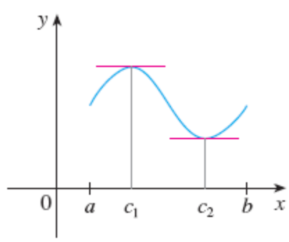
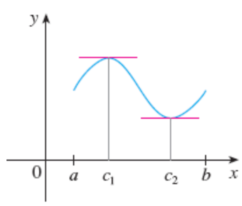


(c)

CASE III: $f(x) < f(a)$ for some x in (a,b)



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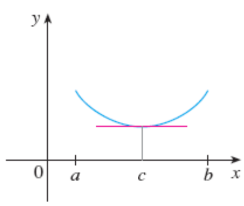

(d)

<u>Example 1</u>: Let $f(x) = x^2 - 4x + 1$; $[0,4]$ Find all the numbers *c* that satisfies the conclusion of Rolle's Theorem.

The Mean Value Theorem

The Mean Value Theorem: Let *f* be a function that satisfies the following hypotheses:

(1) *f* is continuous on the closed interval [*a*,*b*].

(2) *f* is differentiable on the open interval (*a*, *b*).

Then there is a number *c* in (*a*,*b*) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

or, equivalently,

$$f(b) - f(a) = f'(c)\left(b - a\right)$$

<u>Example 3</u>: Find all number *c* that satisfy the conclusion of the MVT.

(1) $f(x) = x^3 - x \quad , [0, 2]$

(2) $f(x) = \sqrt[3]{x} \quad ,[0,1]$

Example 4: Suppose that $f(0) = -3$ and $f'(x) \le 5$ for all x. How large can $f(2)$ possibly be?

Section 3.8: An Application to Economics

Three functions of importance to an economist or a manufacturer are

$C(x)$ = total cost of producing x units of a product during some time period

$R(x)$ = total revenue from selling x units of the product during the time period

$P(x)$ = total profit obtained by selling x units of the product during the time period

These are called, respectively, the ***cost function***, ***revenue function***, and ***profit function***. If all units produced are sold, then these are related by

$$P(x) = R(x) - C(x)$$

Now let's consider marketing. Let $p(x)$ be the price per unit that the company can charge if it sells x units. Then p is called the ***demand function*** (or ***price function***). If x units are sold and the price per unit is $p(x)$, then the total revenue is

$$R(x) = xp(x)$$

Note:

- The marginal cost is the rate of change of C with respect to x. In other word, the marginal cost function is the derivative, $C'(x)$, of the cost function.
- The marginal revenue is the rate of change of R with respect to x. That is the derivative, $R'(x)$, of the revenue function.
- The marginal profit is the rate of change of P with respect to x. That is the derivative, $P'(x)$, of the profit function.

Example 1:

(a) Show that if the profit $P(x)$ is a maximum, then the marginal revenue equals the marginal cost.

(b) If $C(x) = 16,000 + 500x - 1.6x^2 + 0.004x^3$ is the cost function and $p(x) = 1,700 - 7x$ is the demand function, find the production level that will maximize profit.

Example 2: A liquid form of antibiotic manufactured by a pharmaceutical firm is sold in bulk at a price of $200 per unit. If the total production cost (in dollars) for *x* units is

$$C(x) = 500,000 + 80x + 0.003x^2$$

and if the production capacity of the firm is at most 30,000 units in a specified time, how many units of antibiotic must be manufactured and sold in that time to maximize the profit?

Example 3: The manager of 100-unit apartment complex knows from experience that all units will be occupied if the rent is \$800 per month. A market survey suggests that, on average, one additional unit will remain vacant for each \$10 increase in rent. What rent should the manager charge to maximize revenue?