

# EE320 (1/2016)

## INTRODUCTORY MATHEMATICAL ECONOMICS

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### MATRIX ALGEBRA AND ITS APPLICATION

# PART I: BASIC MATRIX ALGEBRA

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# Topics

- Representation of a system of equations by matrix notations
- Review matrices and matrix operation
- Determinant and singularity of matrix
- Matrix inversion by determinant
- Cramer's rule

# Why do economists need matrix algebra?

- Suppose you are asked to solve the following system of simultaneous equations:

$$Q_{d1} = 8 - 3P_1 + P_2 - 4P_3, \quad Q_{s1} = 2 + 2P_1, \quad Q_{d1} = Q_{s1}$$

$$Q_{d2} = 12 + 5P_1 - 2P_2 + 2P_3, \quad Q_{s2} = 2 + 3P_2, \quad Q_{d2} = Q_{s2}$$

$$Q_{d3} = 7 - 6P_1 + P_2 - 3P_3, \quad Q_{s3} = 5 + 4P_3, \quad Q_{d3} = Q_{s3}$$

Question: How would you derive the equilibrium prices and quantities for the 3 goods?

➤ **Linear algebra**: solution by elimination of variables

➔ This could be quite tedious and prone to mistakes.

➤ **Matrix algebra**?

# Why do economists need matrix algebra?

## (Cont'd)

- **Matrix algebra** can be very useful in handling a large system of simultaneous equations, commonly used in economics.
- Why?
  - It provides a **compact way of writing an equation system**.
  - It leads to a way of **testing the existence of a solution** by evaluation of a *determinant*.
  - It gives a **method of finding the solution** (if exists).
- Drawback: Matrix algebra is applicable only to linear-equation system.

# System of Equations in Matrix Form

- Given a system of  $m$  linear equations in  $n$  variables:

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = d_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = d_2$$

.....

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = d_m$$

- We can form a matrix from the above system of equations as:

$AX = d$ , where

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} = [a_{ij}]$$

Coefficients

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix}$$

Variables

$$d = \begin{bmatrix} d_1 \\ d_2 \\ \dots \\ d_m \end{bmatrix}$$

Constant terms

## System of Equations in Matrix Form (Cont'd)

- Example:

$$Q_d = Q_s$$

$$Q_d = a - bP$$

$$Q_s = -c + dP$$

- Write in matrix form  $AX = d$ :

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & b \\ 0 & 1 & -d \end{bmatrix} \quad X = \begin{bmatrix} Q_d \\ Q_s \\ P \end{bmatrix} \quad d = \begin{bmatrix} 0 \\ a \\ -c \end{bmatrix}$$

- Later on we will show that, if a certain condition is satisfied ( $|A| \neq 0$ ), then  $X = A^{-1}d$ , where  $A^{-1}$  is the inverse of matrix  $A$ .

# Example

- Suppose a firm sells 3 products in 3 regions.

| Good   | Sale Volume (Unit: Piece) |         |       |
|--------|---------------------------|---------|-------|
|        | North                     | Central | South |
| Good A | 100                       | 50      | 120   |
| Good B | 80                        | 110     | 60    |
| Good C | 90                        | 70      | 130   |

- Matrix of sale volumes (rows-> product & columns->region) can be written as:

$$Q = \begin{bmatrix} 100 & 50 & 120 \\ 80 & 110 & 60 \\ 90 & 70 & 130 \end{bmatrix}$$

# Types of Matrices

- Let  $A$  be an  $m \times n$  matrix, where  $m$  is the number of rows and  $n$  is the number of columns.

- When  $m=n$ , matrix  $A$  is a **square matrix**.
- When  $n=1$ , matrix  $A$  is a **column vector**.
- When  $m=1$ , matrix  $A$  is a **row vector**.
- An **identity matrix** is a square matrix containing ones along the diagonal and zeros elsewhere:

$$I_n = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

- A **null matrix** is a matrix whose elements are all zero.
- A **diagonal matrix** is a square matrix  $A_{n \times n} = [a_{ij}]$  with  $a_{ij}=0$  when  $i \neq j$ .

# Basic Matrix Operations

- Let  $A$  and  $B$  be  $m \times n$  matrices.
  1.  $A = B$  if and only if  $a_{ij} = b_{ij}$  for all values of  $i$  and  $j$ .
  2. If  $A = [a_{ij}]_{m \times n}$  and  $B = [b_{ij}]_{m \times n}$ , then
$$A + B = [a_{ij}]_{m \times n} + [b_{ij}]_{m \times n} = [a_{ij} + b_{ij}]_{m \times n}$$
$$A - B = [a_{ij}]_{m \times n} - [b_{ij}]_{m \times n} = [a_{ij} - b_{ij}]_{m \times n}$$

Example:

$$\begin{bmatrix} 3 & 1 \\ 4 & 7 \end{bmatrix} + \begin{bmatrix} -5 & 2 \\ 6 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 3 \\ 10 & 7 \end{bmatrix}$$

3. If  $\alpha$  is a real number, then  $\alpha A = \alpha[a_{ij}]_{m \times n} = [\alpha a_{ij}]_{m \times n}$

Example:

$$5 \begin{bmatrix} 3 & 1 \\ 4 & 7 \end{bmatrix} = \begin{bmatrix} 15 & 5 \\ 20 & 35 \end{bmatrix}$$

# Basic Matrix Operations (cont'd)

## 4. Rules of matrix addition and multiplication by scalars

*a)*  $(A + B) + C = A + (B + C)$

*b)*  $A + B = B + A$

*c)*  $A + 0 = A$

*d)*  $A + (-A) = 0$

*e)*  $(\alpha + \beta)A = \alpha A + \beta A$

*f)*  $\alpha (A + B) = \alpha A + \alpha B$

# Matrix Multiplication

- Suppose  $A = [a_{ij}]_{m \times n}$  and  $B = [b_{ij}]_{n \times p}$ .

Then, the product  $C = AB$  is the  $m \times p$  matrix  $C = [c_{ij}]_{m \times p}$ , whose element  $ij$  is the inner product

$$C_{ij} = \sum_{r=1}^n a_{ir} b_{rj} = a_{i1} b_{1j} + a_{i2} b_{2j} + \dots + a_{in} b_{nj}.$$

- Example:

$$\text{Let } A_{1 \times 2} = \begin{bmatrix} a_{11} & a_{12} \end{bmatrix} \quad B_{2 \times 3} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}$$

$$\Rightarrow C = AB = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} & a_{11}b_{13} + a_{12}b_{23} \end{bmatrix}$$

# Example: Matrix Multiplication

- Suppose the sale volume matrix and the price vector are given by:

$$Q = \begin{bmatrix} 100 & 50 & 120 \\ 80 & 110 & 60 \\ 90 & 70 & 130 \end{bmatrix} \quad P = [P_A \quad P_B \quad P_C] = [2 \quad 3 \quad 5]$$

- The total revenue from the sales of all products is:

$$TR = PQ = [2 \quad 3 \quad 5] \begin{bmatrix} 100 & 50 & 120 \\ 80 & 110 & 60 \\ 90 & 70 & 130 \end{bmatrix} = [TR_{North} \quad TR_{Central} \quad TR_{South}]$$

$$TR = [2(100) + 3(80) + 5(90) \quad 2(50) + 3(110) + 5(70) \quad 2(120) + 3(60) + 5(130)]$$

$$TR = [890 \quad 780 \quad 1070]$$

# Rule for Matrix Multiplication:

a)  $AB \neq BA$  (in general)

b)  $(AB)C = A(BC)$

c)  $A(B + C) = AB + AC$

d)  $(B + C)A = BA + CA$

- Example:

Let  $A = \begin{bmatrix} 3 & 1 \\ 4 & 7 \end{bmatrix}$        $B = \begin{bmatrix} -5 & 2 \\ 6 & 0 \end{bmatrix}$

$$\rightarrow AB = \begin{bmatrix} 3 & 1 \\ 4 & 7 \end{bmatrix} \begin{bmatrix} -5 & 2 \\ 6 & 0 \end{bmatrix} = \begin{bmatrix} -15 + 6 & 6 \\ -20 + 42 & 8 \end{bmatrix} = \begin{bmatrix} -9 & 6 \\ 22 & 8 \end{bmatrix}$$

$$\rightarrow BA = \begin{bmatrix} -5 & 2 \\ 6 & 0 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 4 & 7 \end{bmatrix} = \begin{bmatrix} -15 + 8 & -5 + 14 \\ 18 & 6 \end{bmatrix} = \begin{bmatrix} -7 & 9 \\ 18 & 6 \end{bmatrix}$$

# Matrix Transposition

- The transpose of any  $m \times n$  matrix  $A$ , denoted by  $A'$  or  $A^T$ , is defined as the  $n \times m$  matrix whose first column is the first row of  $A$ , and the second column is the second row of  $A$ , and so on.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \longrightarrow A' = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \cdots & \cdots & \cdots & \cdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix}$$

- Example:

$$Q = \begin{bmatrix} 100 & 80 & 90 \\ 50 & 110 & 70 \\ 120 & 60 & 130 \end{bmatrix} \longrightarrow Q' = \begin{bmatrix} 100 & 50 & 120 \\ 80 & 110 & 60 \\ 90 & 70 & 130 \end{bmatrix}$$

# Properties of transposes

$$a) \quad (A')' = A$$

$$b) \quad (A + B)' = A' + B'$$

$$c) \quad (AB)' = B'A'$$

$$d) \quad (\alpha A)' = \alpha A'$$

Note: A square matrix  $A$  is **symmetric** (i.e.  $A = A'$ ) iff  $a_{ij} = a_{ji}$  for all  $i, j$

•Example:

$$A = \begin{bmatrix} 3 & 1 \\ 4 & 7 \end{bmatrix} \quad B = \begin{bmatrix} -5 & 2 \\ 6 & 0 \end{bmatrix}$$

$$\rightarrow (AB)' = \begin{bmatrix} -9 & 6 \\ 22 & 8 \end{bmatrix}' = \begin{bmatrix} -9 & 22 \\ 6 & 8 \end{bmatrix}$$

$$\rightarrow B'A' = \begin{bmatrix} -5 & 6 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 1 & 7 \end{bmatrix} = \begin{bmatrix} -15+6 & -20+42 \\ 6 & 8 \end{bmatrix} = \begin{bmatrix} -9 & 22 \\ 6 & 8 \end{bmatrix}$$

# Matrix Inversion

- The inverse of matrix  $A$ , denoted by  $A^{-1}$ , is defined only if  $A$  is a square matrix and must satisfy the condition

$$AA^{-1} = A^{-1}A = I.$$

- Properties of the inverse:

- a) If  $A^{-1}$  exists, then  $(A^{-1})^{-1} = A$ .
- b) If  $AB$  is invertible, then  $(AB)^{-1} = B^{-1}A^{-1}$
- c)  $(A')^{-1} = (A^{-1})'$
- d)  $(cA)^{-1} = c^{-1}A^{-1}$ , where  $c \neq 0$

- If a square matrix  $A$  has an inverse,  $A$  is said to be nonsingular.

If  $A$  is not invertible, then  $A$  is called a singular matrix.

- The nonsingularity condition is required for a solution of a linear-equation system to exist.

# Inverse Matrix and Solution of Linear-Equation System

- Given the equation system in matrix notation:

$$AX = d$$

$$A^{-1}AX = A^{-1}d$$

$$\Rightarrow X = A^{-1}d$$

where  $X$  is the column vector of variables, and  $A^{-1}d$  is the column vector of solution values.

➤ Since  $A^{-1}$ , if it exists, is unique, the solution  $A^{-1}d$  must be unique values.

- Example (Suppose we know  $A^{-1}$ ):

$$\begin{bmatrix} Q_d \\ Q_s \\ P \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & b \\ 0 & 1 & -d \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ a \\ -c \end{bmatrix} = \frac{-1}{b+d} \begin{bmatrix} -b & -d & -b \\ d & -d & -b \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ a \\ -c \end{bmatrix} = \frac{1}{b+d} \begin{bmatrix} ad - bc \\ ad - bc \\ a + c \end{bmatrix}$$

# How to Find the Inverse of a Matrix

- For 2x2 matrix  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ , we can use the following formula:

$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

- Use Gauss-Jordan elimination to transform  $[A|I]$  into  $[I|A^{-1}]$ .
- Use determinant and adjoint method\*\*:

$$A^{-1} = \frac{1}{\det A} (\text{adjoint of } A) = \frac{1}{\det A} (\text{cofactor of } A)^T$$

# Determinant

- One way to test whether an inverse of a matrix exists (i.e. it is nonsingular) is to use determinant.
- The determinant of a *square* matrix  $A$ , denoted by  $|A|$ , is a uniquely defined associated with that matrix.

- Determinant of order 1:  $|A| = |a| = a$

- Determinant of order 2:  $|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$

- Determinant of order 3:

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

➤  $|A| = a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{31}a_{22})$

## Example: Determinants of Order 3

- Evaluate the following determinant:

$$\begin{aligned} |A| &= \begin{vmatrix} 5 & 7 & 9 \\ 2 & 5 & 6 \\ 9 & 0 & 12 \end{vmatrix} \\ &= 5 \begin{vmatrix} 5 & 6 \\ 0 & 12 \end{vmatrix} - 7 \begin{vmatrix} 2 & 6 \\ 9 & 12 \end{vmatrix} + 9 \begin{vmatrix} 2 & 5 \\ 9 & 0 \end{vmatrix} \\ &= 5(60 - 0) - 7(24 - 54) + 9(0 - 45) \\ &= 300 + 210 - 405 \\ &= 105 \end{aligned}$$

# Evaluating an $n$ th-Order Determinants by Laplace Expansion

Definitions: Let  $A$  be an  $n \times n$  matrix.

➤ The **minor** of the element  $a_{ij}$  is:  $|M_{ij}| =$

$$\begin{vmatrix} a_{11} & \cdots & a_{1j-1} & a_j & a_{1j+1} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & & \vdots & \ddots & \vdots \\ a_{i1} & \cdots & a_{ij-1} & a_{ij} & a_{ij+1} & \cdots & a_{in} \\ \vdots & \ddots & \vdots & & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nj-1} & a_{nj} & a_{nj+1} & \cdots & a_{nn} \end{vmatrix}$$

➤ The **cofactor** of the element  $a_{ij}$  is:  $|C_{ij}| = (-1)^{i+j} |M_{ij}|$

➤ The **determinant** of the matrix  $A$  of order  $n$  can be found by **the Laplace expansion of any row or any column** as follows:

$$|A| = \sum_{j=1}^n a_{ij} |C_{ij}| \quad [\text{expansion by the } i\text{th row}]$$

$$|A| = \sum_{i=1}^n a_{ij} |C_{ij}| \quad [\text{expansion by the } j\text{th column}]$$

# Example: Laplace Expansion

- Evaluate the following determinant by using Laplace expansion:

$$|A| = \begin{vmatrix} 5 & 7 & 9 \\ 2 & 5 & 6 \\ 9 & 0 & 12 \end{vmatrix}$$

# Basic Properties of Determinants (1)

- Property I

The interchange of rows and columns does not affect the value of a determinant. That is,  $|A'| = |A|$ .

Example:

$$|A| = \begin{vmatrix} 3 & 1 \\ 4 & 7 \end{vmatrix} = 17 \qquad |A'| = \begin{vmatrix} 3 & 4 \\ 1 & 7 \end{vmatrix} = 17$$

- Property II

The interchange of any two rows (or columns) will alter the sign, but not the numerical value of the determinant.

Example:

$$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \qquad |B| = \begin{vmatrix} c & d \\ a & b \end{vmatrix} = bc - ad$$

# Basic Properties of Determinants (2)

- Property III

If all element in any one row (or column) is multiplied by a scalar  $k$ , the determinant is multiplied by  $k$ .

Example:  $|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$        $|B| = \begin{vmatrix} a & b \\ kc & kd \end{vmatrix} = kad - kbc = k|A|$

- Property IV

The addition (or subtraction) of a multiple of any row to (from) another row will leave the value of the determinant unaltered.

Example:  $|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$        $|B| = \begin{vmatrix} a & b \\ c + ka & d + kb \end{vmatrix} = ad + kab - bc - kab = |A|$

# Basic Properties of Determinants (3)

- Property V

If one row (or column) is a multiple of another row (or column), the value of the determinant will be zero.

Example:

$$[A] = \begin{bmatrix} 3 & 4 & 2 \\ 15 & 20 & 10 \\ 4 & 0 & 1 \end{bmatrix}$$

→  $|A| = 0$ . Thus, this matrix A is linearly dependent.

Note:

The condition of *linear independence* is a sufficient condition for the *nonsingularity* of a matrix. For the rows (or columns) to be linearly independent, none must be a linear combination of the rest.

# Determinantal Criterion for Nonsingularity

- Summary:

$|A| \neq 0 \iff$  there is row (column) independence in matrix A  
A is nonsingular  
 $A^{-1}$  exists  
a unique solution  $X^* = A^{-1}d$  exists

# Matrix Inversion by Determinant

- Assume that  $A$  is an  $n \times n$  nonsingular matrix, and  $|A| \neq 0$ .
- The inverse of matrix  $A$  is:

$$A^{-1} = \frac{1}{|A|} \text{adj}(A)$$

where

$$\text{adj}(A) \equiv C^T_{n \times n} \equiv \begin{bmatrix} |C_{11}| & |C_{21}| & \cdots & |C_{n1}| \\ |C_{12}| & |C_{22}| & \cdots & |C_{n2}| \\ \cdots & \cdots & \cdots & \cdots \\ |C_{1n}| & |C_{2n}| & \cdots & |C_{nn}| \end{bmatrix}$$

and

$$|C_{ij}| = (-1)^{i+j} \underbrace{\begin{vmatrix} a_{11} & \cdots & a_{1j-1} & a_{1j} & a_{1j+1} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{i1} & \cdots & a_{iy-1} & a_{ij} & a_{iy+1} & \cdots & a_{in} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nj-1} & a_{nj} & a_{nj+1} & \cdots & a_{nn} \end{vmatrix}}_{|M_{ij}|}$$

# Steps to Find the Inverse of a Matrix A

1. Find the determinant  $|A|$ . If  $|A| = 0$ , then the inverse does not exist.
2. Find the **cofactors** of all the elements of A, and arrange them as a matrix:  $C = [|C_{ij}|]$
3. Take the transpose of the cofactor matrix, C, to get the ***adj(A)***.
4. Divide the adj(A) by the determinant.

**Example:**

Find the inverse of  $A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & b \\ 0 & 1 & -d \end{bmatrix}$

**Answer:**

$$A^{-1} = \frac{1}{-(b+d)} \begin{bmatrix} -b & -d & -b \\ d & -d & -b \\ 1 & -1 & 1 \end{bmatrix}$$

# Example

- Find the inverse of the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & 4 \\ 0 & 1 & -13 \end{bmatrix}$$

# Finding Solutions Using Inverse Matrix

- For a problem  $AX = d$ , if  $|A| \neq 0$ , then  $X = A^{-1}d$ .
- Example:

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & b \\ 0 & 1 & -d \end{bmatrix} \quad X = \begin{bmatrix} Q_d \\ Q_s \\ P \end{bmatrix} \quad d = \begin{bmatrix} 0 \\ a \\ -c \end{bmatrix}$$

- Solution:

$$X^* = \begin{bmatrix} Q_d^* \\ Q_s^* \\ P^* \end{bmatrix} = A^{-1}d = \frac{1}{-(b+d)} \begin{bmatrix} -b & -d & -b \\ d & -d & -b \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ a \\ -c \end{bmatrix} = \frac{1}{b+d} \begin{bmatrix} ad - bc \\ ad - bc \\ a + c \end{bmatrix}$$

# Cramer's Rule

- Given an equation system  $Ax = d$ , where  $A$  is an  $n \times n$  nonsingular matrix, the solution value of the  $j$ th variable can be obtained from:

$$x_j^* = \frac{|A_j|}{|A|} = \frac{1}{|A|} \begin{vmatrix} a_{11} & a_{12} & \dots & d_1 & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & d_2 & \dots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & d_n & \dots & a_{nn} \end{vmatrix}$$

where  $|A_j|$  is the determinant of the matrix  $A$  when the  $j$ th column is replaced by the constant terms  $d_1 \dots d_n$ .

## Example: Cramer's Rule

- Given the system of two equations:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix}$$

- By Cramer's rule, the solutions are given by:

$$x^* = \frac{\begin{vmatrix} u & b \\ v & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} = \frac{ud - bv}{ad - bc} \quad \text{and} \quad y^* = \frac{\begin{vmatrix} a & u \\ c & v \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} = \frac{av - uc}{ad - bc}$$

# Another Example: Market Equilibrium

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & b \\ 0 & 1 & -d \end{bmatrix} \quad X = \begin{bmatrix} Q_d \\ Q_s \\ P \end{bmatrix} \quad d = \begin{bmatrix} 0 \\ a \\ -c \end{bmatrix}$$

➤ Using Cramer's rule, we get:

$$Q^* = \frac{\begin{vmatrix} 0 & -1 & 0 \\ a & 0 & b \\ -c & 1 & -d \end{vmatrix}}{\begin{vmatrix} 1 & -1 & 0 \\ 1 & 0 & b \\ 0 & 1 & -d \end{vmatrix}} = \frac{ad - bc}{b + d} \quad \text{and} \quad P^* = \frac{\begin{vmatrix} 1 & -1 & 0 \\ 1 & 0 & a \\ 0 & 1 & -c \end{vmatrix}}{\begin{vmatrix} 1 & -1 & 0 \\ 1 & 0 & b \\ 0 & 1 & -d \end{vmatrix}} = \frac{a + c}{b + d}$$

# MATRIX ALGEBRA: APPLICATIONS

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# Topics

- Partial and general market equilibrium
- Excise tax and market equilibrium
- Simple macroeconomic model
- IS-LM model
- Input-output model

# Partial Market Equilibrium (1)

- Given the system of equations

$$Q_d = 24 - 4P \quad \text{-- (1)}$$

$$Q_s = -27 + 13P \quad \text{-- (2)}$$

$$Q_d = Q_s \quad \text{-- (3)}$$

- Write the system in matrix form  $AX = d$ :

$$A = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & -13 \\ 1 & -1 & 0 \end{bmatrix} \quad X = \begin{bmatrix} Q_d \\ Q_s \\ P \end{bmatrix} \quad d = \begin{bmatrix} 24 \\ -27 \\ 0 \end{bmatrix}$$

# Partial Market Equilibrium (2)

- Using Cramer's rule to find  $Q^*$  and  $P^*$

➤ Answer:  $P^* = 3$ ,  $Q^* = 12$

# General Market Equilibrium (1)

- Market for good 1

$$Q_{d1} - Q_{s1} = 0 \quad (1)$$

$$Q_{d1} = a_0 + a_1P_1 + a_2P_2 \quad (2)$$

$$Q_{s1} = b_0 + b_1P_1 + b_2P_2 \quad (3)$$

- Market for good 2:

$$Q_{d2} - Q_{s2} = 0 \quad (4)$$

$$Q_{d2} = \alpha_0 + \alpha_1P_1 + \alpha_2P_2 \quad (5)$$

$$Q_{s2} = \beta_0 + \beta_1P_1 + \beta_2P_2 \quad (6)$$

- Write the above system of equations in the matrix form:

Define  $c_i \equiv a_i - b_i$  and  $\gamma_i \equiv \alpha_i - \beta_i$

$$\begin{bmatrix} c_1 & c_2 \\ \gamma_1 & \gamma_2 \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \begin{bmatrix} -c_0 \\ -\gamma_0 \end{bmatrix}$$

# General Market Equilibrium (2)

- Find the equilibrium prices and quantities for good 1 and good 2.

Define:  $c_i \equiv a_i - b_i$

$$\gamma_i \equiv \alpha_i - \beta_i$$

➤ Answer:

$$P_1^* = \frac{c_2 \gamma_0 - c_0 \gamma_2}{c_1 \gamma_2 - c_2 \gamma_1}$$

$$P_2^* = \frac{c_0 \gamma_1 - c_1 \gamma_0}{c_1 \gamma_2 - c_2 \gamma_1}$$

# Excise Tax and Market Equilibrium: Specific Tax (1)

- Suppose a specific tax t baht per unit is imposed on consumer.

$$Q_D = Q_S \quad (1)$$

$$Q_D = a - b(P + T) \quad (2)$$

$$Q_S = -c + dP \quad (3)$$

- Write the above system of the equations in the matrix form:

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & b \\ 0 & 1 & -d \end{bmatrix}$$

$$X = \begin{bmatrix} Q_d \\ Q_s \\ P \end{bmatrix}$$

$$d = \begin{bmatrix} 0 \\ a - bT \\ -c \end{bmatrix}$$

# Excise Tax and Market Equilibrium: Specific Tax (2)

- Find the new equilibrium prices and quantity.
- Answer:

$$P^* = \frac{a + c + dT}{b + d} = P'_D$$

$$P^T = P'_D - T = \frac{a + c - bT}{b + d} = P'_S$$

$$Q^* = \frac{ad - bc - bdT}{b + d}$$

# Excise Tax and Market Equilibrium: Ad Varolem Tax (1)

- Suppose an  $t\%$  tax is imposed on consumer.

$$Q_D = Q_S \quad (1)$$

$$Q_D = a - bP^T \quad (2)$$

$$Q_S = -c + dP \quad (3)$$

$$P^T = (1+t)P \quad (4)$$

- Write the above system of the equations in the matrix form:

$$A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & 0 & b \\ 0 & 1 & -d & 0 \\ 0 & 0 & -(1+t) & 1 \end{bmatrix} \quad X = \begin{bmatrix} Q_d \\ Q_s \\ P \\ P^T \end{bmatrix} \quad d = \begin{bmatrix} 0 \\ a \\ -c \\ 0 \end{bmatrix}$$

# Excise Tax and Market Equilibrium: Ad Varolem Tax (2)

- Find the new equilibrium prices and quantity.
- Answer:

$$P^* = \frac{a + c}{b(1 + t) + d} = P'_S$$

$$P^t = (1 + t)P'_S = \frac{(1 + t)(a + c)}{b(1 + t) + d} = P'_D$$

$$Q^* = \frac{ad - (1 + t)bc}{b(1 + t) + d}$$

# Simple Macroeconomic Model (1)

- Keynesian model:

$$Y = C + I_0 + G_0 + X_0 - M$$

$$C = a + bY_d$$

$$M = mY_d$$

$$Y_d = Y - T$$

$$T = tY$$

- Reduce the above system into:

$$Y = C + I_0 + G_0 + X_0 - m(1-t)Y$$

$$C = a + b(1-t)Y$$

$$\rightarrow [1 - m(1-t)]Y - C = I_0 + G_0 + X_0$$

$$-b(1-t)Y + C = a$$

## Simple Macroeconomic Model (2)

- Write the above system in the matrix form, and find the equilibrium national income.

$$\begin{bmatrix} 1 + m(1 - t) & -1 \\ -b(1 - t) & 1 \end{bmatrix} \begin{bmatrix} Y \\ C \end{bmatrix} = \begin{bmatrix} I_0 + G_0 + X_0 \\ a \end{bmatrix}$$

- Solutions:**

$$Y^* = \frac{a + I_0 + G_0 + X_0}{1 - b(1 - t) + m(1 - t)}$$

$$C^* = \frac{a[1 + m(1 - t)] + b(1 - t)(I_0 + G_0 + X_0)}{1 - b(1 - t) + m(1 - t)}$$

# IS-LM Model (1)

- Commodity Market:

$$Y = C + I + G_0$$

$$C = a + bY_d \quad (a > 0, 0 < b < 1)$$

$$I = I_0 - ir \quad (I_0, i > 0)$$

$$Y_d = Y - T \quad \text{where } T = tY \quad (0 < t < 1)$$

$$\Rightarrow Y = a + b(1-t)Y + I_0 - ir + G_0 \quad : \text{IS}$$

- Money Market:

$$M^S = M_0$$

$$M^D = kY - hr \quad (k, h > 0)$$

$$\Rightarrow M_0 = kY - hr \quad : \text{LM}$$

- Write a matrix form of the above IS-LM equations:

## IS-LM Model (2)

- Use Cramer's rule to find the equilibrium national income and interest rate.

$$\begin{bmatrix} 1 - b(1 - t) & i \\ k & -h \end{bmatrix} \begin{bmatrix} Y \\ r \end{bmatrix} = \begin{bmatrix} a + I_0 + G_0 \\ M_0 \end{bmatrix}$$

- Solutions:

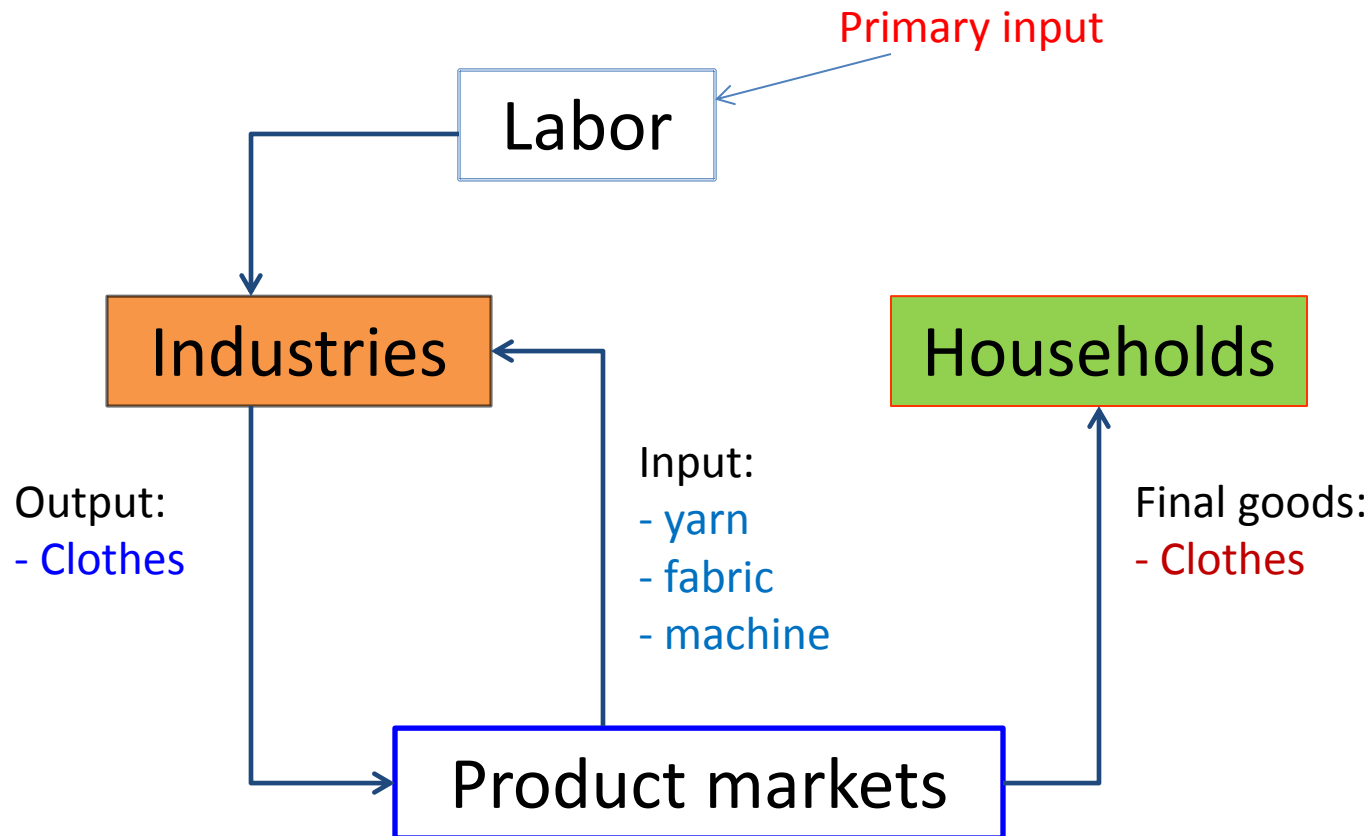
$$Y^* = \frac{(a + I_0 + G_0)h + iM_0}{ik + h[1 - b(1 - t)]}$$

$$r^* = \frac{(a + I_0 + G_0)k - [1 - b(1 - t)]M_0}{ik + h[1 - b(1 - t)]}$$

# Leontif Input-Output Model (1)

- This model characterizes input-output relationships among  $n$  interlinked industries in an economy.
- It seeks to determine the output level that each of the  $n$  industries should produce to meet the total demand for that product.
- **Applications:** Use in production planning, e.g. planning for economic development of a country
- **Assumptions:**
  - Each industry produces only one homogenous commodity.
  - Each industry uses a fixed input ratio for the production of its output.
  - Production in every industry is subject to constant returns to scale.

# Leontief Input-Output *Open* Model (2)



# Leontif Input-Output Model (3)

- Consider an economy with  $n$  industries.
  - All the outputs are produced need to meet (i) the **input demand** of the same  $n$  industries and (ii) the **final demand** (e.g. consumer demand)
  - 2 types of inputs: (i) **intermediate inputs** and (ii) **primary inputs** (e.g. labor)
- Notations:
  - $x_i$  = total number of units of good  $i$  that industry  $i$  is going to produce
  - $a_{ij}$  = the number of units of good  $i$  needed to produce one unit of good  $j$  ( $a_{ij}$  = input coefficient)
  - $a_{ij}x_j$  = the number of units of good  $i$  needed to produce  $x_j$  unit of good  $j$
  - $d_i$  = final demand for good  $i$

# Leontif Input-Output Model (4)

- Equilibrium between supply and demand for each good  $i$ :

$$x_i = a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n + d_i$$

- The system of equations for  $n$  industries:

$$x_1 = a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n + d_1$$

$$x_2 = a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n + d_2$$

.....

$$x_n = a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n + d_n$$

- Input-coefficient matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

# Leontif Input-Output Model (4)

- Rearranging the above system of equations:

$$\begin{array}{rcl}
 (1 - a_{11})x_1 - & a_{12}x_2 - \dots - & a_{1n}x_n = d_1 \\
 -a_{21}x_1 & + (1 - a_{22})x_2 - \dots - & a_{2n}x_n = d_2 \\
 \dots\dots\dots & & \\
 -a_{n1}x_1 - & a_{n2}x_2 - \dots & + (1 - a_{nn})x_n = d_n
 \end{array}$$

- Leontif model in matrix form:

$$\underbrace{\begin{bmatrix} (1 - a_{11}) & -a_{12} & \dots & -a_{1n} \\ -a_{21} & (1 - a_{22}) & \dots & -a_{2n} \\ \vdots & \vdots & & \vdots \\ -a_{n1} & -a_{n2} & \dots & (1 - a_{nn}) \end{bmatrix}}_{\text{Leontif matrix}} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{bmatrix} \quad \text{or} \quad (I - A)x = d$$

- Given that  $(I - A)^{-1}$  exists, the unique solution of the system is:

$$x^* = (I - A)^{-1}d$$

# Leontif Input-Output Model (5)

- A numerical example:

| Industry<br>( $i$ ) | Input Demand for Output $j$ |             |         | Final<br>Demand<br>(\$) |
|---------------------|-----------------------------|-------------|---------|-------------------------|
|                     | Agriculture                 | Manufacture | Service |                         |
| Agriculture $x_1$   | 0.2                         | 0.3         | 0.2     | 10                      |
| Manufacture $x_2$   | 0.4                         | 0.1         | 0.2     | 5                       |
| Service $x_3$       | 0.1                         | 0.3         | 0.2     | 6                       |

- Questions:
  - What level of output should each industry produce?
  - Find the input coefficients of the primary input (labor) for each output, and determine the required amount of the primary input needed in this economy.

# Leontif Input-Output Model (6)

- Write the input-coefficient matrix:

$$A = \begin{bmatrix} 0.2 & 0.3 & 0.2 \\ 0.4 & 0.1 & 0.2 \\ 0.1 & 0.3 & 0.2 \end{bmatrix}$$

- The input-output system in the matrix form  $(I-A)x = d$ :

$$\begin{bmatrix} 0.8 & -0.3 & -0.2 \\ -0.4 & 0.9 & -0.2 \\ -0.1 & -0.3 & 0.8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 5 \\ 6 \end{bmatrix}$$

# Leontif Input-Output Model (7)

- Given that 
$$\begin{bmatrix} 0.8 & -0.3 & -0.2 \\ -0.4 & 0.9 & -0.2 \\ -0.1 & -0.3 & 0.8 \end{bmatrix}^{-1} = \frac{1}{0.38} \begin{bmatrix} 0.66 & 0.3 & 0.24 \\ 0.34 & 0.62 & 0.24 \\ 0.21 & 0.27 & 0.6 \end{bmatrix}$$

- The solutions for the output levels to be produced are:

$$\begin{bmatrix} x_1^* \\ x_2^* \\ x_2^* \end{bmatrix} = \frac{1}{0.384} \begin{bmatrix} 0.66 & 0.3 & 0.24 \\ 0.34 & 0.62 & 0.24 \\ 0.21 & 0.27 & 0.6 \end{bmatrix} \begin{bmatrix} 10 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 24.84 \\ 20.68 \\ 18.36 \end{bmatrix}$$

- The level of primary input (labor) required for this economy:

$$a_{01} = 0.3 ; \quad a_{02} = 0.3 ; \quad a_{03} = 0.4$$

$$\rightarrow a_0 = 0.3(24.84) + 0.3(20.68) + 0.4(18.36) = 21$$