

EE320 (2/2013)

INTRODUCTORY MATHEMATICAL ECONOMICS

MATRIX ALGEBRA AND ITS APPLICATION

PART I: BASIC MATRIX ALGEBRA

Topics

- Representation of a system of equations by matrix notations
- Review matrices and matrix operation
- Determinant and singularity of matrix
- Matrix inversion by determinant
- Cramer's rule

Why do economists need matrix algebra?

- Suppose you are asked to solve the following system of simultaneous equations:

$$\begin{aligned}Q_{d1} &= 8 - 3P_1 + P_2 - 4P_3, & Q_{s1} &= 2 + 2P_1, & Q_{d1} &= Q_{s1} \\Q_{d2} &= 12 + 5P_1 - 2P_2 + 2P_3, & Q_{s2} &= 2 + 3P_2, & Q_{d2} &= Q_{s2} \\Q_{d3} &= 7 - 6P_1 + P_2 - 3P_3, & Q_{s3} &= 5 + 4P_3, & Q_{d3} &= Q_{s3}\end{aligned}$$

Question: How would you derive the equilibrium prices and quantities for the 3 goods?

- **Linear algebra**: solution by elimination of variables
 - ➔ This could be quite tedious and prone to mistakes.
- **Matrix algebra**?

Why do economists need matrix algebra? (Cont'd)

- **Matrix algebra** can be very useful in handling a large system of simultaneous equations, commonly used in economics.
- Why?
 - It provides a **compact way of writing an equation system**.
 - It leads to a way of **testing the existence of a solution** by evaluation of a *determinant*.
 - It gives a **method of finding the solution** (if exists).
- Drawback: Matrix algebra is applicable only to linear-equation system.

System of Equations in Matrix Form

- Given a system of m linear equations in n variables:

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = d_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = d_2$$

.....

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = d_m$$

- We can form a matrix from the above system of equations as:

$AX = d$, where

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} = [a_{ij}] \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix} \quad d = \begin{bmatrix} d_1 \\ d_2 \\ \dots \\ d_m \end{bmatrix}$$

Coefficients

Variables

Constant terms

System of Equations in Matrix Form (Cont'd)

- Example:

$$Q_d = Q_s$$

$$Q_d = a - bP$$

$$Q_s = -c + dP$$

→ Write in matrix form $AX = d$:

- Later on we will show that, if a certain condition is satisfied ($|A| \neq 0$), then $X = A^{-1}d$, where A^{-1} is the inverse of matrix A .

Example

- Suppose a firm sells 3 products in 3 regions.

Good	Sale Volume (Unit: Piece)		
	North	Central	South
Good A	100	50	120
Good B	80	110	60
Good C	90	70	130

- Matrix of sale volumes (rows-> product & columns->region) can be written as:

$$Q = \begin{bmatrix} 100 & 50 & 120 \\ 80 & 110 & 70 \\ 90 & 70 & 130 \end{bmatrix}$$

Types of Matrices

- Let A be an $m \times n$ matrix, where m is the number of rows and n is the number of columns.

1. When $m=n$, matrix A is a **square matrix**.
2. When $n=1$, matrix A is a **column vector**.
3. When $m=1$, matrix A is a **row vector**.
4. An **identity matrix** is a square matrix containing ones along the diagonal and zeros elsewhere:

$$I_n = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

5. A **null matrix** is a matrix whose elements are all zero.
6. A **diagonal matrix** is a square matrix $A_{n \times n} = [a_{ij}]$ with $a_{ij}=0$ when $i \neq j$.

Basic Matrix Operations

- Let A and B be $m \times n$ matrices.
 1. $A = B$ if and only if $a_{ij} = b_{ij}$ for all values of i and j .
 2. If $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$, then
$$A + B = [a_{ij}]_{m \times n} + [b_{ij}]_{m \times n} = [a_{ij} + b_{ij}]_{m \times n}$$
$$A - B = [a_{ij}]_{m \times n} - [b_{ij}]_{m \times n} = [a_{ij} - b_{ij}]_{m \times n}$$

Example:

$$\begin{bmatrix} 3 & 1 \\ 4 & 7 \end{bmatrix} + \begin{bmatrix} -5 & 2 \\ 6 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 3 \\ 10 & 7 \end{bmatrix}$$

3. If α is a real number, then $\alpha A = \alpha [a_{ij}]_{m \times n} = [\alpha a_{ij}]_{m \times n}$

Example:

$$5 \begin{bmatrix} 3 & 1 \\ 4 & 7 \end{bmatrix} = \begin{bmatrix} 15 & 5 \\ 20 & 35 \end{bmatrix}$$

Basic Matrix Operations (cont'd)

4. Rules of matrix addition and multiplication by scalars

$$a) \quad (A + B) + C = A + (B + C)$$

$$b) \quad A + B = B + A$$

$$c) \quad A + 0 = A$$

$$d) \quad A + (-A) = 0$$

$$e) \quad (\alpha + \beta)A = \alpha A + \beta A$$

$$f) \quad \alpha(A + B) = \alpha A + \alpha B$$

Matrix Multiplication

- Suppose $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{n \times p}$.

Then, the product $C = AB$ is the $n \times p$ matrix $C = [c_{ij}]_{n \times p}$, whose element ij is the inner product

$$C_{ij} = \sum_{r=1}^n a_{ir} b_{rj} = a_{i1} b_{1j} + a_{i2} b_{2j} + \dots + a_{in} b_{nj}.$$

- Example:

$$\text{Let } A_{1 \times 2} = \begin{bmatrix} a_{11} & a_{12} \end{bmatrix} \quad B_{2 \times 3} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}$$

$$\rightarrow C = AB = \begin{bmatrix} a_{11} b_{11} + a_{12} b_{21} & a_{11} b_{12} + a_{12} b_{22} & a_{11} b_{13} + a_{12} b_{23} \end{bmatrix}$$

Example: Matrix Multiplication

- Suppose the sale volume matrix and the price vector are given by:

$$Q = \begin{bmatrix} 100 & 50 & 120 \\ 80 & 110 & 70 \\ 90 & 70 & 130 \end{bmatrix} \quad P = [P_A \quad P_B \quad P_C] = [2 \quad 3 \quad 5]$$

- The total revenue from the sales of all products is:

$$TR = PQ = [2 \quad 3 \quad 5] \begin{bmatrix} 100 & 50 & 120 \\ 80 & 110 & 70 \\ 90 & 70 & 130 \end{bmatrix} = [TR_{North} \quad TR_{Central} \quad TR_{South}]$$

$$TR = [2(100) + 3(80) + 5(90) \quad 2(50) + 3(110) + 5(70) \quad 2(120) + 3(60) + 5(130)]$$

$$TR =$$

Rule for Matrix Multiplication:

a) $AB \neq BA$ (in general)

b) $(AB)C = A(BC)$

c) $(A + B)C = AC + BC$

d) $(B + C)A = BA + CA$

- Example:

Let $A = \begin{bmatrix} 3 & 1 \\ 4 & 7 \end{bmatrix}$ $B = \begin{bmatrix} -5 & 2 \\ 6 & 0 \end{bmatrix}$

→ $AB =$

→ $BA =$

Matrix Transposition

- The transpose of any $m \times n$ matrix A , denoted by A' or A^T , is defined as the $n \times m$ matrix whose first column is the first row of A , and the second column is the second row of A , and so on.

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \longrightarrow A' = \begin{bmatrix} a_{11} & a_{21} & \dots & a_{m1} \\ a_{12} & a_{22} & \dots & a_{m2} \\ \dots & \dots & \dots & \dots \\ a_{1n} & a_{2n} & \dots & a_{mn} \end{bmatrix}$$

- Example:

$$Q = \begin{bmatrix} 100 & 80 & 90 \\ 50 & 110 & 70 \\ 120 & 60 & 130 \end{bmatrix} \longrightarrow$$

Properties of transposes

$$a) (A')' = A$$

$$b) (A + B)' = A' + B'$$

$$c) (AB)' = B'A'$$

$$d) (\alpha A)' = \alpha A'$$

Note: A square matrix A is **symmetric** (i.e. $A = A'$) iff $a_{ij} = a_{ji}$ for all i, j

• Example:

$$A = \begin{bmatrix} 3 & 1 \\ 4 & 7 \end{bmatrix} \quad B = \begin{bmatrix} -5 & 2 \\ 6 & 0 \end{bmatrix}$$

$$\rightarrow (AB)' =$$

$$\rightarrow B'A' =$$

Matrix Inversion

- The inverse of matrix A , denoted by A^{-1} , is defined only if A is a square matrix and must satisfy the condition

$$AA^{-1} = A^{-1}A = I.$$

- Properties of the inverse:
 - a) If A^{-1} exists, then $(A^{-1})^{-1} = A$.
 - b) If AB is invertible, then $(AB)^{-1} = B^{-1}A^{-1}$
 - c) $(A')^{-1} = (A^{-1})'$
 - d) $(cA)^{-1} = c^{-1}A^{-1}$, where $c \neq 0$
- If a square matrix A has an inverse, A is said to be nonsingular. If A is not invertible, then A is called a singular matrix.
 - The nonsingularity condition is required for a solution of a linear-equation system to exist.

Inverse Matrix and Solution of Linear-Equation System

- Given the equation system in matrix notation:

$$AX = d$$

$$A^{-1}AX = A^{-1}d$$

$$\rightarrow X = A^{-1}d$$

where X is the column vector of variables, and $A^{-1}d$ is the column vector of solution values.

➤ Since A^{-1} , if it exists, is unique, the solution $A^{-1}d$ must be unique values.

- Example:

- Next, we will study how to determine A^{-1} by determinant.

Determinant

- One way to test whether an inverse of a matrix exists (i.e. it is nonsingular) is to use determinant.
- The determinant of a square matrix A , denoted by $|A|$, is a uniquely defined associated with that matrix.

- Determinant of order 1: $|A| = |a| = a$

- Determinant of order 2: $|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$

- Determinant of order 3:

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

➤ $|A| = a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{31}a_{22})$

Example: Determinants of Order 3

- Evaluate the following determinant:

$$|A| = \begin{vmatrix} 5 & 7 & 9 \\ 2 & 5 & 6 \\ 9 & 0 & 12 \end{vmatrix}$$

Evaluating an n th-Order Determinants by Laplace Expansion

Definitions: Let A be an $n \times n$ matrix.

➤ The **minor** of the element a_{ij} is: $|M_{ij}| =$

$$\begin{vmatrix} a_{11} & \cdots & a_{1j-1} & a_{1j} & a_{1j+1} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{i1} & \cdots & a_{ij-1} & a_{ij} & a_{ij+1} & \cdots & a_{in} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nj-1} & a_{nj} & a_{nj+1} & \cdots & a_{nn} \end{vmatrix}$$

➤ The **cofactor** of the element a_{ij} is: $|C_{ij}| = (-1)^{i+j} |M_{ij}|$

➤ The **determinant** of the matrix A of order n can be found by **the Laplace expansion of any row or any column** as follows:

$$|A| = \sum_{j=1}^n a_{ij} |C_{ij}| \quad [\text{expansion by the } i\text{th row}]$$

$$|A| = \sum_{i=1}^n a_{ij} |C_{ij}| \quad [\text{expansion by the } j\text{th column}]$$

Basic Properties of Determinants (1)

- Property I

The interchange of rows and columns does not affect the value of a determinant. That is, $|A'| = |A|$.

Example:

$$|A| = \begin{vmatrix} 3 & 1 \\ 4 & 7 \end{vmatrix} = 17 \quad |A'| = \begin{vmatrix} 3 & 4 \\ 1 & 7 \end{vmatrix} = 17$$

- Property II

The interchange of any two rows (or columns) will alter the sign, but not the numerical value of the determinant.

Example:

$$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \quad |B| = \begin{vmatrix} c & d \\ a & b \end{vmatrix} = bc - ad$$

Basic Properties of Determinants (2)

- Property III

If all element in any one row (or column) is multiplied by a scalar k , the determinant is multiplied by k .

Example: $|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$ $|B| = \begin{vmatrix} a & b \\ kc & kd \end{vmatrix} = kad - kbc = k|A|$

- Property IV

The addition (or subtraction) of a multiple of any row to (from) another row will leave the value of the determinant unaltered.

Example: $|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$ $|B| = \begin{vmatrix} a & b \\ c + ka & d + kb \end{vmatrix} = ad + kab - bc - kab = |A|$

Basic Properties of Determinants (3)

- Property V

If one row (or column) is a multiple of another row (or column), the value of the determinant will be zero.

Example:

$$[A] = \begin{bmatrix} 3 & 4 & 2 \\ 15 & 20 & 10 \\ 4 & 0 & 1 \end{bmatrix}$$

→ $|A| = 0$. Thus, this matrix A is linearly dependent.

Note:

The condition of *linear independence* is a sufficient condition for the *nonsingularity* of a matrix. For the rows (or columns) to be linearly independent, none must be a linear combination of the rest.

Determinantal Criterion for Nonsingularity

- Summary:

$|A| \neq 0 \iff$ there is row (column) independence in matrix A
A is nonsingular
 A^{-1} exists
a unique solution $X^* = A^{-1}d$ exists

Matrix Inversion by Determinant

- Assume that A is an $n \times n$ nonsingular matrix, and $|A| \neq 0$.
- The inverse of matrix A is:

$$A^{-1} = \frac{1}{|A|} \text{adj}(A)$$

where

$$\text{adj}(A) \equiv C'_{n \times n} \equiv \begin{bmatrix} |C_{11}| & |C_{21}| & \cdots & |C_{n1}| \\ |C_{12}| & |C_{22}| & \cdots & |C_{n2}| \\ \cdots & \cdots & \cdots & \cdots \\ |C_{1n}| & |C_{2n}| & \cdots & |C_{nn}| \end{bmatrix} \quad \text{and} \quad |C_{ij}| = (-1)^{i+j} \begin{vmatrix} a_{11} & \cdots & a_{1j-1} & a_j & a_{1j+1} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{i1} & \cdots & a_{ij-1} & a_{ij} & a_{ij+1} & \cdots & a_{in} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nj-1} & a_{nj} & a_{nj+1} & \cdots & a_{nn} \end{vmatrix}$$

Steps to Find the Inverse of a Matrix A

1. Find the determinant $|A|$. If $|A| = 0$, then the inverse does not exist.
2. Find the **cofactors** of all the elements of A, and arrange them as a matrix: $C = [|C_{ij}|]$
3. Take the transpose of the cofactor matrix, C, to get the ***adj(A)***.
4. Divide the $\text{adj}(A)$ by the determinant.

Example:

Find the inverse of $A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & b \\ 0 & 1 & -d \end{bmatrix}$

Answer:

Finding Solutions Using Inverse Matrix

- For a problem $AX = d$, if $|A| \neq 0$, then $X = A^{-1}d$.
- Example:

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & b \\ 0 & 1 & -d \end{bmatrix} \quad X = \begin{bmatrix} Q_d \\ Q_s \\ P \end{bmatrix} \quad d = \begin{bmatrix} 0 \\ a \\ -c \end{bmatrix}$$

- Solution:

Cramer's Rule

- Given an equation system $Ax = d$, where A is an $n \times n$ nonsingular matrix, the solution value of the j th variable can be obtained from:

$$x_j^* = \frac{|A_j|}{|A|} = \frac{1}{|A|} \begin{vmatrix} a_{11} & a_{12} & \dots & d_1 & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & d_2 & \dots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & d_n & \dots & a_{nn} \end{vmatrix}$$

where $|A_j|$ is the determinant of the matrix A when the j th column is replaced by the constant terms $d_1 \dots d_n$.

Example: Cramer's Rule

- Given the system of two equations:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix}$$

- By Cramer's rule, the solutions are given by:

Another Example: Market Equilibrium

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & b \\ 0 & 1 & -d \end{bmatrix} \quad X = \begin{bmatrix} Q_d \\ Q_s \\ P \end{bmatrix} \quad d = \begin{bmatrix} 0 \\ a \\ -c \end{bmatrix}$$

➤ Using Cramer's rule, we get:

MATRIX ALGEBRA: APPLICATIONS

Topics

- Partial and general market equilibrium
- Excise tax and market equilibrium
- Simple macroeconomic model
- IS-LM model
- Input-output model

Partial Market Equilibrium (1)

- Given the system of equations

$$Q_d = 24 - 4P \quad \text{-- (1)}$$

$$Q_s = -27 + 13P \quad \text{-- (2)}$$

$$Q_d = Q_s \quad \text{-- (3)}$$

- Write the system in matrix form $AX = d$:

Partial Market Equilibrium (2)

- Using Cramer's rule to find Q^* and P^*
- **Answer:** $P^* = 3$, $Q^* = 12$

General Market Equilibrium (1)

- Market for good 1

$$Q_{d1} - Q_{s1} = 0 \quad (1)$$

$$Q_{d1} = a_0 + a_1P_1 + a_2P_2 \quad (2)$$

$$Q_{s1} = b_0 + b_1P_1 + b_2P_2 \quad (3)$$

- Market for good 2:

$$Q_{d2} - Q_{s2} = 0 \quad (4)$$

$$Q_{d2} = \alpha_0 + \alpha_1P_1 + \alpha_2P_2 \quad (5)$$

$$Q_{s2} = \beta_0 + \beta_1P_1 + \beta_2P_2 \quad (6)$$

- Write the above system of equations in the matrix form:

Define $c_i \equiv a_i - b_i$ and $\gamma_i \equiv \alpha_i - \beta_i$

General Market Equilibrium (2)

- Find the equilibrium prices and quantities for good 1 and good 2.

Define: $c_i \equiv a_i - b_i$

$$\gamma_i \equiv \alpha_i - \beta_i$$

➤ Answer:

Excise Tax and Market Equilibrium: Specific Tax (1)

- Suppose a specific tax t baht per unit is imposed on consumer.

$$Q_D = Q_S \quad (1)$$

$$Q_D = a - b(P + T) \quad (2)$$

$$Q_S = -c + dP \quad (3)$$

- Write the above system of the equations in the matrix form:

Excise Tax and Market Equilibrium: Specific Tax (2)

- Find the new equilibrium prices and quantity.
- **Answer:**

Excise Tax and Market Equilibrium: Ad Varolem Tax (1)

- Suppose an $t\%$ tax is imposed on consumer.

$$Q_D = Q_S \quad (1)$$

$$Q_D = a - bP^T \quad (2)$$

$$Q_S = -c + dP \quad (3)$$

$$P^T = (1+t)P \quad (4)$$

- Write the above system of the equations in the matrix form:

Excise Tax and Market Equilibrium: Ad Valorem Tax (2)

- Find the new equilibrium prices and quantity.

Simple Macroeconomic Model (1)

- Keynesian model:

$$Y = C + I_0 + G_0 + X_0 - M$$

$$C = a + bY_d$$

$$M = mY_d$$

$$Y_d = Y - T$$

$$T = tY$$

- Reduce the above system into:

Simple Macroeconomic Model (2)

- Write the above system in the matrix form, and find the equilibrium national income.

IS-LM Model (1)

- Commodity Market:

$$Y = C + I + G_0$$

$$C = a + bY_d \quad (a > 0, 0 < b < 1)$$

$$I = I_0 - ir \quad (I_0, i > 0)$$

$$Y_d = Y - T \quad \text{where } T = tY \quad (0 < t < 1)$$

$$\rightarrow Y = a + b(1-t)Y + I_0 - ir + G_0 \quad : \text{IS}$$

- Money Market:

$$M^S = M_0$$

$$M^D = kY - hr \quad (k, h > 0)$$

$$\rightarrow M_0 = kY - hr \quad : \text{LM}$$

- Write a matrix form of the above IS-LM equations:

IS-LM Model (2)

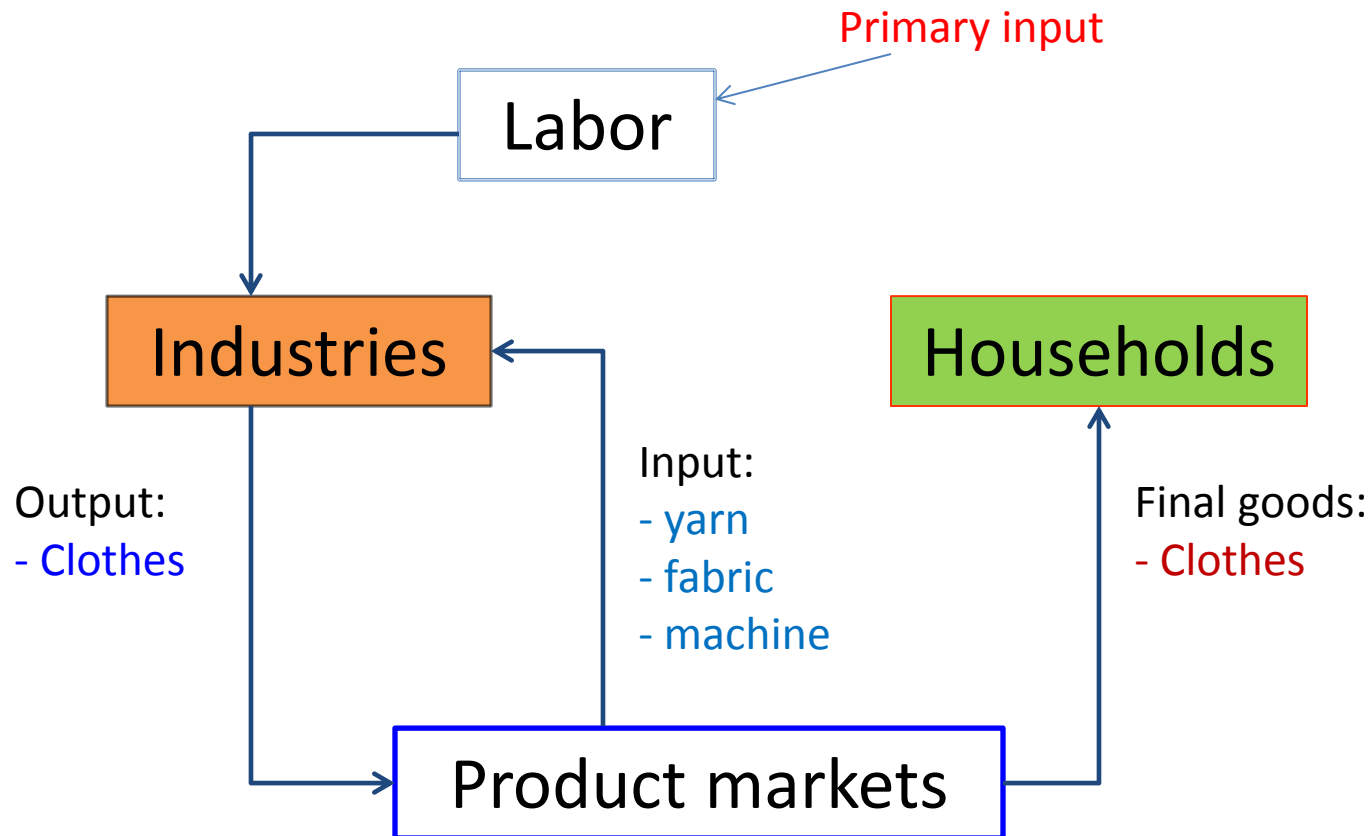
- Use Cramer's rule to find the equilibrium national income and interest rate.

$$\begin{bmatrix} 1-b(1-t) & i \\ k & -h \end{bmatrix} \begin{bmatrix} Y \\ r \end{bmatrix} = \begin{bmatrix} a + I_0 + G_0 \\ M_0 \end{bmatrix}$$

Leontif Input-Output Model (1)

- This model characterizes input-output relationships among n interlinked industries in an economy.
- It seeks to determine **the output level that each of the n industries should produce to meet the total demand for that product.**
- **Applications:** Use in **production planning**, e.g. planning for economic development of a country
- **Assumptions:**
 - Each industry produces only one **homogenous commodity**.
 - Each industry uses a **fixed input ratio** for the production of its output.
 - Production in every industry is subject to **constant returns to scale**.

Leontief Input-Output *Open* Model (2)



Leontif Input-Output Model (3)

- Consider an economy with n industries.
 - All the outputs are produced need to meet (i) the **input demand** of the same n industries and (ii) the **final demand** (e.g. consumer demand)
 - 2 types of inputs: (i) **intermediate inputs** and (ii) **primary inputs** (e.g. labor)
- Notations:
 - x_i = total number of units of good i that industry i is going to produce
 - a_{ij} = the number of units of good i needed to produce one unit of good j
(a_{ij} = input coefficient)
 - $a_{ij}x_j$ = the number of units of good i needed to produce x_j unit of good j
 - d_i = final demand for good i

Leontif Input-Output Model (4)

- Equilibrium between supply and demand for each good i :

$$x_i = a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n + d_i$$

- The system of equations for n industries:

$$x_1 = a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n + d_1$$

$$x_2 = a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n + d_2$$

.....

$$x_n = a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n + d_n$$

- Input-coefficient matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

Leontif Input-Output Model (4)

- Rearranging the above system of equations:

$$\begin{array}{rcl}
 (1 - a_{11})x_1 - & a_{12}x_2 - \dots - & a_{1n}x_n = d_1 \\
 - a_{21}x_1 & + (1 - a_{22})x_2 - \dots - & a_{2n}x_n = d_2 \\
 \dots\dots\dots & & \\
 - a_{n1}x_1 - & a_{n2}x_2 - \dots & + (1 - a_{nn})x_n = d_n
 \end{array}$$

- Leontif model in matrix form:

$$\underbrace{\begin{bmatrix} (1 - a_{11}) & -a_{12} & \dots & -a_{1n} \\ -a_{21} & (1 - a_{22}) & \dots & -a_{2n} \\ \vdots & \vdots & & \vdots \\ -a_{n1} & -a_{n2} & \dots & (1 - a_{nn}) \end{bmatrix}}_{\text{Leontif matrix}} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{bmatrix} \quad \text{or} \quad (I - A)x = d$$

- Given that $(I - A)^{-1}$ exists, the unique solution of the system is:

$$x^* = (I - A)^{-1}d$$

Leontif Input-Output Model (5)

- A numerical example:

Industry	Input Demand			Final Demand (\$)
	Agriculture x_1	Manufacture x_2	Service x_3	
Agriculture	0.2	0.3	0.2	10
Manufacture	0.4	0.1	0.2	5
Service	0.1	0.3	0.2	6

- Questions:
 - What level of output should each industry produce?
 - Find the input coefficients of the primary input (labor) for each output, and determine the required amount of the primary input needed in this economy.

Leontif Input-Output Model (6)

- Write the input-coefficient matrix:

$$A = \begin{bmatrix} 0.2 & 0.3 & 0.2 \\ 0.4 & 0.1 & 0.2 \\ 0.1 & 0.3 & 0.1 \end{bmatrix}$$

- The input-output system in the matrix form $(I-A)x = d$:

Leontif Input-Output Model (7)

- Given that
$$\begin{bmatrix} 0.8 & -0.3 & -0.2 \\ -0.4 & 0.9 & -0.2 \\ -0.1 & -0.3 & 0.8 \end{bmatrix}^{-1} = \frac{1}{0.38} \begin{bmatrix} 0.66 & 0.3 & 0.24 \\ 0.34 & 0.62 & 0.24 \\ 0.21 & 0.27 & 0.6 \end{bmatrix}$$

- The solutions for the output levels to be produced are:

$$\begin{bmatrix} x_1^* \\ x_2^* \\ x_2^* \end{bmatrix} = \frac{1}{0.384} \begin{bmatrix} 0.66 & 0.3 & 0.24 \\ 0.34 & 0.62 & 0.24 \\ 0.21 & 0.27 & 0.6 \end{bmatrix} \begin{bmatrix} 10 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 24.84 \\ 20.68 \\ 18.36 \end{bmatrix}$$

- The level of primary input (labor) required for this economy:

$$a_{01} = 0.3 ; \quad a_{02} = 0.3 ; \quad a_{01} = 0.4$$

