Exercise 4

Application of the Derivative: Related Rates, Linearization & Differentials

1 Linearization & Differentials

1. Determine the linear approximation for $\sin(\theta)$ at $\theta = 0$.
2. Find a linearization of the given function at the indicated number.
   (a) $f(x) = \ln(x)$ at $x = 1$.
   (b) $f(x) = x + e^{x-5}$ at $x = 5$
3. Find an approximation of each given quantity by using a local linear approximation.
   (a) $(8.1)^{2/3}$
   (b) $\frac{(-0.9)^3}{1+(-0.9)^2}$
4. Compute the differential dy for $y = x^2 \sin(2x)$.
5. Compute dy and $\triangle y$ for $y = \frac{1}{\sqrt{x}}$ as x changes 4 from to 4.02 .
6. Consider the function
$$f(x) = \frac{1}{(1+2x)^3}.$$
   (i) Find a local linear approximation of f at $x = 0$.
   (ii) Use (i) to approximate $(1.1)^{-3}$.
7. The area of a circle with radius r is $A = \pi r^2$.
   (i) If the radius of a circle changes from 4 cm to 5 cm, find the exact change in the area.
   (ii) By using the *approximation from differentials*, what is the approximate change in the area? Determine the *relative error* of this approximation.

2 Related Rates

1. Let $s(t) = 2\sqrt{t} + \cos(\pi t)$ be the position function of an object that moves on a horizontal line. Find the velocity and acceleration functions.
2. Let $s(t) = 2t^3 - 12t^2 + 48$ be the position function of an object that moves on a horizontal line.
   (a) Find the velocity function $v(t)$ and the acceleration function $a(t)$.
   (b) Find the position of the object when $a(t) = 0$.
3. A tank in the shape of a right circular cylinder of radius 8 m is being filled with water at a constant rate of 10 m^3/min. How fast is the level of the water rising? (Or at what rate is the height of the water increasing?)

4. A plate in the shape of an equilateral triangle expands with time. The length of a side increases at a constant rate of 2 cm/hr. At what rate is the area increasing when the side is 8 cm?

5. Ship S_1 sails north and passes a dock with a constant rate of 10 miles per hour at noon. Ship S_2 sails west and passes the same dock at a constant rate of 15 miles per hour at 1:00pm. At what rate is the distance between the two ships changing at 2:00pm?

Additional Problems (Related Rates)

1. A cube is expanding with time. How is the rate at which the volume increases related to the rate at which the length of a side increases?

2. The volume of a rectangular box is $V = xyz$. Given that each side expands at a constant rate of 10 cm/min, find the rate at which the volume is expanding when $x = 1$ cm, $y = 2$ cm, and $z = 3$ cm.

3. A plane at an altitude of 4 km passes directly over a tracking telescope on the ground. When the angle of the elevation is $60°$, it is observed that this angle is decreasing at a rate of 30 deg/min. How fast is the plane travelling?

4. A rocket is traveling at a constant rate of 1000 mi/h with a constant angle of $60°$ to the horizontal ground.
   (a) At what rate is its altitude increasing?
   (b) What is the ground speed of the rocket ?

5. An oil tank in the shape of a right cylinder of radius 8 is being filled at a constant rate of 10 m^3/min. How fast is the level of the oil rising?

6. Water leaks out the bottom of the conical tank at a constant rate of 1 ft^3/min.

   (a) At what rate is the level of the water changing when the water is 6 ft deep?

   (b) At what rate is the radius of the water changing when the water i 6 ft deep?

   (c) Assume the tank was full at $t = 0$. At what rate is the radius of the water changing at $t = 6$ min?

7. Assume that a cube of ice melts in such a manner that it always retinas its cubical shape. If the volume of the cube decreases at a rate of 0.25 in^3/min, how fast is the surface area of the cube changing when the surface area is 54 in^2?

8. A 15-ft ladder is leaning against a wall of a house. The bottom of the ladder is pulled away from the base of the wall at a constant rate of 2 ft/min. At what rate is the top of the ladder sliding down the wall at the instant when the bottom of the ladder is 5 ft from the wall?

9. Sand flows from the top half of the conical hourglass at a constant rate of 4 cm^3/s. Express the rate at which the height of the bottom pile increases in terms of the height of the sand.

10. A pulley is secured to the edge of a dock that is 15 ft above the surface of the water. A small boat is being pulled toward the dock by means of a rope on the pulley. The rope is attached to the bow o the boat 3 ft above the water line. If the rope is puu;ed in at a constant rate of 1 ft/s, how fast does the boat approach the dock when it is 16 ft from the dock?

11. At 8 A.M. ship S_1 is 20 km due north of ship S_2. Ship S_1 sails south at a rate of 9 km/h and ship S_2 sails west at a rate o 12 km/h. At 9:20 A.M., at what rate is the distance between the two ships changing ?

12. A stone dropped into a still pond causes a circular wave. Assume that the radius of the wave expands at a constant rate of 2 ft/s.

    (a) How fast does the diameter of the circular wave increase?

    (b) How fast does the circumference of the circular wave increase?

    (c) How fast does the area of the circular wave expand when the radius is 3 ft?

    (d) How fast does the area of the circular wave expand when the area is 8π ft^2 ?