Non-linearities and Time-Varying Volatility Models

Autoregressive Conditional Heteroscedasticity ARCH(1) Model

ARCH(q) Model

ARCH in Mean (ARCH-M) Model

Generalized ARCH or GARCH(p,q) Model

Asymmetric GARCH Models

- GJR or Threshold GARCH (TARCH) Model
- Exponential GARCH (EGARCH) Model

ARCH(1) Model

$$y_t = \beta' x_t + \varepsilon_t$$

$$\varepsilon_t = u_t \sqrt{\alpha_0 + \alpha_1 \varepsilon_{t-1}^2}$$

where $E\left[\varepsilon_t \middle| x_t, \varepsilon_{t-1}\right] = 0$ then $E\left[\varepsilon_t \middle| x_t\right] = 0$

Classical regression model follows OLS assumptions (homoscedasticity).

However, the conditional variance is:

$$Var\left[\varepsilon_t \middle| \varepsilon_{t-1}\right] = E\left[\varepsilon_t^2 \middle| \varepsilon_{t-1}\right] = E\left[u_t^2\right]\left[\alpha_0 + \alpha_1 \varepsilon_{t-1}^2\right]$$

$$= \alpha_0 + \alpha_1 \varepsilon_{t-1}^2$$

ARCH(1) Model

Since classical regression model follows OLS assumptions, OLS estimators are efficient. However, with the conditional heteroscedasticity, nonlinear estimation method (such as MLE) can provide more efficient estimators.

Conditional log-likelihood of the model:

$$\log L = -\frac{T}{2}\log(2\pi) - \frac{1}{2}\sum_{t=1}^{T}\log(\alpha_0 + \alpha_1\varepsilon_{t-1}^2) - \frac{1}{2}\sum_{t=1}^{t}\frac{\varepsilon_t^2}{\alpha_0 + \alpha_1\varepsilon_{t-1}^2}$$

where $\varepsilon_t = y_t - \beta' x_t$

ARCH(q) Model

$$y_t = \beta' x_t + \varepsilon_t$$

ARCH(q) process:

$$\sigma_t^2 = \alpha_0 + \alpha_1 \varepsilon_{t-1}^2 + \alpha_2 \varepsilon_{t-2}^2 + \cdots + \alpha_q \varepsilon_{t-q}^2$$

ARCH-in-Mean Model

$$y_t = \beta' x_t + \delta \sigma_t^2 + \varepsilon_t$$

ARCH process – Variance Equation:

$$\sigma_t^2 = \alpha_0 + \alpha_1 \varepsilon_{t-1}^2 + \alpha_2 \varepsilon_{t-2}^2 + \cdots + \alpha_q \varepsilon_{t-q}^2$$

Test for ARCH Effects

Hypothesis: H_0: $\alpha_1 = \alpha_2 = \ldots = \alpha_q = 0$

1. Estimate $y_t = \beta' x_t + \varepsilon_t$ obtain $\hat{\varepsilon}_t$

2. Estimate

$$\hat{\varepsilon}_t^2 = \alpha_0 + \alpha_1 \hat{\varepsilon}_{t-1}^2 + \alpha_2 \hat{\varepsilon}_{t-2}^2 + \cdots + \alpha_q \hat{\varepsilon}_{t-q}^2 + u_t$$

obtain R^2

3. Calculate $TR^2 \sim \chi^2(q)$

GARCH(p,q) Model

$$y_t = \beta' x_t + \varepsilon_t$$

GARCH(p,q) process:

$$\sigma_t^2 = \alpha_0 + \delta_1 \sigma_{t-1}^2 + \delta_2 \sigma_{t-2}^2 + \cdots + \delta_p \sigma_{t-p}^2$$

$$+ \alpha_1 \varepsilon_{t-1}^2 + \alpha_2 \varepsilon_{t-2}^2 + \cdots + \alpha_q \varepsilon_{t-q}^2$$

$$= \alpha_0 + \sum_{j=1}^{p} \delta_j \sigma_{t-j}^2 + \sum_{i=1}^{q} \alpha_i \varepsilon_{t-i}^2$$

Asymmetric GARCH Model

Glosten, Jagannathan & Runkle (GJR) or Threshold GARCH (TARCH) Model

TARCH(p,q,r) $$y_t = \beta' x_t + \varepsilon_t$$

Conditional variance process:

$$\sigma_t^2 = \alpha_0 + \sum_{j=1}^{p} \delta_j \sigma_{t-j}^2 + \sum_{i=1}^{q} \alpha_i \varepsilon_{t-i}^2 + \sum_{k=1}^{r} \gamma_k \varepsilon_{t-k}^2 I_{t-k}$$

where $I_{t-1} = 1$ if $\varepsilon_{t-1} < 0$

$= 0$ otherwise

Asymmetric GARCH Model

Exponential GARCH (EGARCH) Model

EGARCH(p,q,r) $y_t = \beta' x_t + \varepsilon_t$

Conditional variance process:

$$\ln(\sigma_t^2) = \alpha_0 + \sum_{j=1}^{p} \delta_j \ln(\sigma_{t-j}^2) + \sum_{i=1}^{q} \alpha_i \left| \frac{\varepsilon_{t-i}}{\sigma_{t-i}} \right| + \sum_{k=1}^{r} \gamma_k \frac{\varepsilon_{t-k}}{\sigma_{t-k}}$$