B.E. International Program

Faculty of Economics, Thammasat University

EE 320 Introductory Mathematical Economics (Section 046402)

Semester 1/2013

Practice Problem 7

(Derivatives of More-Than-One Independent Variable Function) [1]

1. The demand for money, M, in the United States for the period 1929-1952 has been estimated as

$$M = 0.14Y + 76.03(r-2)^{-0.84}, \quad (r > 2)$$

where Y is the annual national income, and r is the interest rate measured in percent per year. Find $\partial M/\partial Y$ and $\partial M/\partial r$ and discuss their signs.

Ans. $\partial M/\partial Y = 0.14 > 0$: Money demand increases with income.

$\partial M/\partial r = -0.84 \cdot 76.03(r-2)^{-1.84} = -63.87(r-2)^{-1.84} < 0$: Money demand decreases as the interest rate increases.

2. The demand for a product depends on the price p of the product and on the price q charged by a competing producer

$$D(p,q) = a - bpq^{-\alpha}$$

where a, b, and α are positive constants with α < 1. Find $D'_p(p,q)$ and $D'_q(p,q)$, and comment on the signs of the partial derivatives.

Ans. $D'_p(p,q) = bq^{-\alpha} < 0$, showing that the demand decreases as its own price increases.

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[1] Questions 1-9 are from Sydsaeter and Hammond, 2008. Questions 10-11 are from Wainwright.

$D_q'(p,q) = -b\alpha p q^{-\alpha-1} > 0$, showing that the demand increases as the price of a competing product increases.

3. Let *x* and *y* be the populations of two cities and *d* the distance between them. Suppose that the number of travelers *T* between the cities is given by

$$T = kxy/d^n \qquad \text{(k and n are positive constants.)}$$

Find $\partial T/\partial x$, $\partial T/\partial y$, and $\partial T/\partial d$, and discuss their signs.

Ans. $\partial T/\partial x = \frac{ky}{d^n} > 0$ and $\partial T/\partial y = \frac{kx}{d^n} > 0$, suggesting that the number of travelers increases as the size of either city increases.

$\partial T/\partial d = -\frac{nkxy}{d^{n+1}} < 0$, suggesting that the number of travelers decreases as the distance between the two cities increases.

4. Let *D(p, m)* indicate a typical consumer's demand for a particular commodity, as a function of its price *p* and the consumer's own income *m*. Show that the proportion *pD/m* of income spent on the commodity increases with income if $\varepsilon_m > 1$ (in which case the good is a "luxury" whereas it is a "necessity" if $\varepsilon_m < 1$).

Ans. $\frac{\partial}{\partial m}\left(\frac{pD(p,m)}{m}\right) = p\frac{mD_m - D}{m^2} = \frac{pD}{m^2}\left[m\frac{\partial D}{\partial m}\cdot\frac{1}{D} - 1\right] = \frac{pD}{m^2}[\varepsilon_m - 1] > 0 \; iff \; \varepsilon_m > 1.$

So, *pD/m* increases with *m* if $\varepsilon_m > 1$ ("luxury" goods).

Similarly, one can show that $\frac{\partial}{\partial m}\left(\frac{pD(p,m)}{m}\right) = \frac{pD}{m^2}[\varepsilon_m - 1] < 0 \; iff \; \varepsilon_m < 1$ ("necessity" goods).

5. The annual herring catch is given by the function $Y(K,S) = 0.06157K^{1.356}S^{0.562}$ of the catching effort (K) and the herring stock (S).

(a) Find $\partial Y/\partial K$ and $\partial Y/\partial S$.

Ans. a) $\partial Y/\partial K \approx 0.083K^{0.356}L^{0.562}$ and $\partial Y/\partial S \approx 0.035K^{1.356}L^{-0.438}$.

(b) If K and S are both doubled, what happens to the catch?

Ans. If K and S are both doubled, the catch becomes $2^{1.356+0.562} = 2^{1.918} \approx 3.779$ times higher.

6. Suppose that a firm produces $Q = f(L) = \sqrt{L}$ units of commodity using L units of labor. Assume that *f'(L) > 0* and *f''(L) < 0*, so f is strictly increasing and strictly concave.

(a) If the firm gets P baht per unit produced and pays w baht for a unit of labor, write down the profit function, and find the first-order condition for profit maximization at $L^* > 0$.

Ans. Profit function: $(L) = Pf(L) - wL$.

F.O.C. $Pf'(L) - w = 0$ ➔ $\frac{P}{2\sqrt{L^*}} = w$ ➔ $L^* = \frac{P^2}{4w^2}$.

(b) Suppose the profit function is replaced by $\pi(L) = Pf(L) - C(L, w)$, where $C(L, w)$ is the cost function. State the first-order condition for profit maximization. By implicit differentiation of the first-order condition, examine how changes in P and w influence the optimal choice of L* (i.e. find $\partial L^*/\partial P$ and $\partial L^*/\partial w$).

Ans. F.O.C : $Pf'(L^*) - C_L'[L^*, w] = 0$.

Define $F \equiv Pf'(L^*) - C_L'[L^*, w] = 0$. Them apply the implicit function rule to get:

$$\partial L^*/\partial P = \frac{f'(L^*)}{C_{LL}''(L^*,w) - pf''(L^*)} \;;\; \partial L^*./\partial w = \frac{C_{Lw}''(L^*,w)}{pf''(L^*) - C_{LL}''(L^*,w)}$$

7. Suppose production X depends on the number of workers N according the formula

$$X = Ng(\frac{\varphi(N)}{N})$$

where g and φ are given differentiable functions. Find expressions for dX/dN and d^2X/dN^2.

Ans. $\frac{dX}{dN} = g(u) + Ng'(u)\frac{du}{dN} = g(u) + g'(u)(\varphi'(N) - u)$, where $u = \frac{\varphi(N)}{N}$

$$\frac{d^2X}{dN^2} = g'(u)\frac{du}{dN} + g''(u)\frac{du}{dN}(\varphi'(N) - u) + g'(u)\left(\varphi''(N) - \frac{du}{dN}\right)$$

$$= \frac{1}{n}g''\left(\frac{\varphi(N)}{N}\right)\left[\varphi'(N) - \frac{\varphi(N)}{N}\right]^2 + g'\left(\frac{\varphi(N)}{N}\right)\varphi''(N).$$

8. Find the marginal rate of substitution (MRS) between y and x when:

(a) $U(x, y) = 2x^{0.4}y^{0.6}$ Ans. $MRS_{yx} = \frac{U_x}{U_y} = \frac{2y}{3x}$

(b) $U(x, y) = xy + y$ Ans. $MRS_{yx} = \frac{U_x}{U_y} = \frac{y}{x+1}$

(c) $U(x, y) = 10(x^{-2} + y^{-2})^{-4}$ Ans. $MRS_{yx} = \frac{U_x}{U_y} = \left(\frac{y}{x}\right)^3$

9. An equilibrium model of labor demand and output pricing leads to the following system of equations:

$$pF'(L) - w = 0$$

$$pF(L) - wL - B = 0 \qquad (*)$$

Suppose that F is twice differentiable with $F'(L) > 0$ and $F''(L) < 0$, and all the variables are positive. Treat w and B as exogenous, so that p and L are endogenous variables which are functions of w and B.

(a) Find expressions for $\partial p/\partial w$, $\partial p/\partial B$, $\partial L/\partial w$, and $\partial L/\partial B$ by implicit differentiation.

Ans. $\partial p/\partial w = \frac{L}{F(L)}$; $\partial p/\partial B = \frac{1}{F(L)}$;

$\partial L/\partial w = \frac{F(L)-LF'(L)}{pF(L)F''(L)}$; $\partial L/\partial B = -\frac{F'(L)}{pF(L)F''(L)}$

Derivations of the above expressions:

$$p \cdot F'(L) - w = 0 \quad \text{— (1)}$$

$$pF(L) - wL - B = 0 \quad \text{— (2)}$$

$$(1) \Rightarrow \quad p \cdot F''(L) \cdot dL + F'(L)\,dp = dw \quad \text{— (3)}$$

$$(2) \Rightarrow \quad \underbrace{p \cdot F'(L)}_{=w}\,dL + F(L)\,dp - w\,dL - L\,dw = dB$$

$$w\,dL + F(L)\,dp - w\,dL - L\,dw = dB$$

$$F(L)\,dp - L\,dw = dB$$

$$\Rightarrow dp = \frac{dB + L\,dw}{F(L)} \quad \text{— (4)}$$

Note: $L = L(w, B)$

Sub dp in (1):

$$p \cdot F''(L)\,dL + F'(L) \cdot \frac{[dB + L\,dw]}{F(L)} = dw$$

$$pF''(L) \cdot F(L)\,dL + F'(L)\,dB + L \cdot F'(L)\,dw = F(L)\,dw$$

$$pF''(L) \cdot F(L) \cdot dL = [-LF'(L) + F(L)]\,dw - F'(L)\,dB$$

$$dL = \frac{[-LF'(L) + F(L)]}{p \cdot F''(L)F(L)}\,dw - \frac{F'(L)}{p \cdot F''(L) \cdot F(L)} \cdot dB$$

$$= L_w = \frac{\partial L}{\partial w} \qquad = L_B = \frac{\partial L}{\partial B}$$

Recall: $dL = L_w \cdot dw + L_B \cdot dB$

Thus, $\boxed{\dfrac{\partial L}{\partial w} = \dfrac{F(L) - LF'(L)}{p \cdot F''(L) \cdot F(L)}}$; $\boxed{\dfrac{\partial L}{\partial B} = \dfrac{-F'(L)}{p \cdot F''(L) \cdot F(L)}}$

To find $\frac{\partial p}{\partial w}$ and $\frac{\partial p}{\partial B}$, recall that $p = p(w, B)$. So,

$$dp = \frac{\partial p}{\partial w} \cdot dw + \frac{\partial p}{\partial B} \cdot dB.$$

From (4), we have $dp = \underbrace{\dfrac{L}{F(L)}}_{p_w} dw + \underbrace{\dfrac{1}{F(L)}}_{p_B} dB.$

Thus, $\boxed{\dfrac{\partial p}{\partial w} = \dfrac{L}{F(L)}}$ and $\boxed{\dfrac{\partial p}{\partial B} = \dfrac{1}{F(L)}}$

(b) What can be said about the signs of these partial derivatives? Show that $\partial L/\partial w < 0$.

Ans. Since $p > 0$, $F'(L) > 0$, and $F''(L) < 0$, $\partial p/\partial w > 0$; $\partial p/\partial B > 0$; $\partial L/\partial B > 0$.

We can show that $F(L) - LF'(L) = B/p > 0$. Thus, $\partial L/\partial w < 0$.

10. If $z = x^3y^2 + x^2y^4 - 3xy$ and $x = r + 3s$ and $y = 2r - s$, then determine $\frac{\partial z}{\partial s}$ when r = 1 and s = 0.

Ans.

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = (3x^2y^2 + 2xy^4 - 3y)(3) + (2x^3y + 4x^2y^3 - 3x)(-1)$$

At r = 1 and s = 0, x = 1, y =2. $\frac{\partial z}{\partial s} = 126 - 31 = 95$

11. If $x^2 + xy + yz + x^2 = 6$, then:

(a) Find $\frac{\partial z}{\partial y}$

Ans. $\frac{\partial z}{\partial y} = -\frac{x+z}{y}$

(b) Evaluate $\frac{\partial z}{\partial y}$ at x =1, y=2, z =1.

Ans. $\frac{\partial z}{\partial y} = -\frac{x+z}{y} = -\frac{2}{2} = -1$