Chapter 4: Linear Transformation

1 General Linear Transformation

Definition 1.1. If $T : V \to W$ is a function from a vector space V into a vector space W, then T is called a **linear transformation** from V to W if, for all vectors $\mathbf{u}$ and $\mathbf{v}$ in V and all scalars k, the following two properties hold.

(i) $T(k\mathbf{u}) = kT(\mathbf{u})$ [**Homogeneity proper**]

(ii) $T(\mathbf{v} + \mathbf{u}) = T(\mathbf{v}) + T(\mathbf{u})$ [**Additivity property**]

In the special case where $V = W$, the linear transformation T is called a**linear operator** on the vector space V.

Remark: The homogeneity and additivity properties of a linear transformation $T : V \to W$ can be used in combination to show that if $\mathbf{v}_1, \ldots, \mathbf{v}_r$ are vectors in V and $k_1, \ldots, k_r$ are any scalars, then

$$T(k_1\mathbf{v}_1 + \cdots + k_r\mathbf{v}_r) = k_1T(\mathbf{v}_1) + \cdots k_r + T(\mathbf{v}_r).$$

Theorem 1.1. If $T : V \to W$ is a linear transformation, then:

(a) $T(\mathbf{0}) = \mathbf{0}$,

(b) $T(\mathbf{u} - \mathbf{v}) = T(\mathbf{u}) - T(\mathbf{v})$ for all $\mathbf{u}.\mathbf{v} \in V$.

I.e.

$$T(-\mathbf{v}) = -\mathbf{v}$$

Example 1.1. Consider **matrix transformations** $T_{\mathbf{A}}$ from $\mathbb{R}^n$ to $\mathbb{R}^m$ defined by

$$\mathbf{T}(\mathbf{x}) = \mathbf{A}\mathbf{x}$$

where $\mathbf{A} \in \mathbb{R}^{m\times n}$. Show that $T_{\mathbf{A}}$ is a linear transformation.

Example 1.2. Consider **zero transformations** T from $\mathbb{R}^n$ to $\mathbb{R}^m$, defined by

$$\mathbf{T}(\mathbf{x}) = \mathbf{0}.$$

Example 1.3. (**Exercise**) Let V be any vector space. The mapping $I : V \to V$ defined by $I(\mathbf{v}) = \mathbf{v}$ is called the **identity operator** on V. Verify that I is linear.

Example 1.4. If V is a vector space and k is any scalar, then the mapping $T : V \to V$ given by $T(\mathbf{x}) = k\mathbf{x}$ is a linear operator on V.

Remarks:
- If $k \in (0, 1)$, then T is called the *contraction* of V with factor k, and
- if $k > 1$, it is called the *dilation* of V with factor k.

Example 1.5. Determine whether the following transformation $T : \mathbb{R}^n \to \mathbf{R}^n$ is linear.

$$T(\mathbf{x}) = \mathbf{x} + \mathbf{x}_0, \qquad \text{for a fixed } x_0 \in \mathbb{R}^n$$

Example 1.6. Let $\mathbf{p} = p(x) = c_0 + c_1 x + \cdots + c_n x^n$ be a polynomial in P_n, and define the transformation $T : P_n \to P_{n+1}$ by

$$T(\mathbf{p}) = T(p(x)) = xp(x).$$

Show that this transformation is linear.

Example 1.7. Let M_{nn} be the vector space of $n \times n$ matrices. In each part determine whether the transformation is linear.

- $T_1(\mathbf{A}) = \mathbf{A}^T$
- $T_2(\mathbf{A}) = \det(\mathbf{A})$

Example 1.8. Let V be a subspace of $F(-\infty, \infty)$, let $x_1, x_2, \ldots, x_n$ be distinct real numbers, and let $T : V \to \mathbb{R}^n$ be the transformation

$$T(f) = (f(x_1), f(x_2), \ldots, f(x_n))$$

that associates with f the n-tuple of function values at $x_1, x_2, \ldots, x_n$. We call this the **evaluation transformation** on V at $x_1, x_2, \ldots, x_n$.

(a) Suppose $f(x) = x^2 - 1$ and $x_1 = -1, x_2 = 2, x_3 = 4$. Find $T(f)$.

(b) Show that T is a linear transformation.

2 Finding Linear Transformations from Images of Basis Vectors

2.1 Matrix Transformation

In this section, we will consider a matrix transformation: $T_A : \mathbb{R}^n \to \mathbb{R}^m$ defied by

$$T_A(\mathbf{x}) = \mathbf{A}\mathbf{x}.$$

Theorem 2.1. If $T_A : \mathbb{R}^n \to \mathbb{R}^m$ and $T_B : \mathbb{R}^n \to \mathbb{R}^m$ are matrix transformations, and if $T_A(\mathbf{x}) = T_B(\mathbf{x})$ for every vector $\mathbf{x}$ in $\mathbb{R}^n$, then $\mathbf{A} = \mathbf{B}$

This theorem is significant because it tells us that there is a one-to-one correspondence between $m \times n$ matrices and matrix transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$ in the sense that every $m \times n$ matrix $\mathbf{A}$ produces exactly one matrix transformation (multiplication by $\mathbf{A}$) and every matrix transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ arises from exactly one $m \times n$ matrix; we call that matrix the **standard matrix for the transformation**.

Consider the **standard basis** of $\mathbb{R}^n$:

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \cdots, \quad \mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

where $\mathbf{e}_i$ is the i-column of identity matrix $\mathbf{I}_n \in \mathbb{R}^{n \times n}$

The **standard matrix** for a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is given by the formula

$$\mathbf{A} = [T(\mathbf{e}_1) \mid T(\mathbf{e}_2) \mid \cdots \mid T(\mathbf{e}_n)]$$

Example 2.1. Find the standard matrix $\mathbf{A}$ for the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by the formula

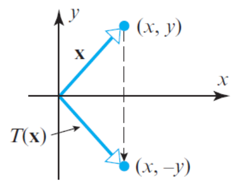
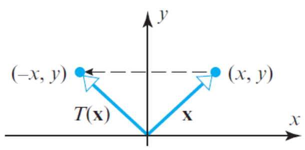
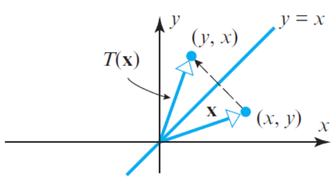
$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 + x_2 \\ x_1 - 3x_2 \\ -x_1 + x_2 \end{bmatrix}.$$

Note: This can be written as $T(x_1, x_2) = (2x_1 + x_2,\ x_1 - 3x_2,\ -x_1 + x_2)$.

2.1.1 Reflection

Some of the most basic matrix operators on $\mathbb{R}^2$ and $\mathbb{R}^3$ are those that map each point into its symmetric image about a fixed line or a fixed plane that contains the origin; these are called **reflection operators**.

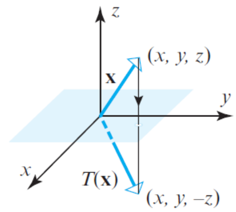
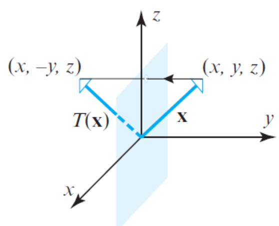
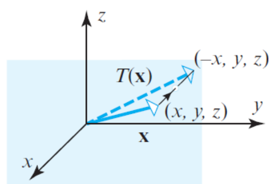
The following table shows the standard matrices for the reflections about the coordinate axes in $\mathbb{R}^2$,

Operator	Illustration	Images of $\mathbf{e}_1$ and $\mathbf{e}_2$	Standard Matrix
Reflection about the x-axis $T(x, y) = (x, -y)$		$T(\mathbf{e}_1) = T(1, 0) = (1, 0)$ $T(\mathbf{e}_2) = T(0, 1) = (0, -1)$	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
Reflection about the y-axis $T(x, y) = (-x, y)$		$T(\mathbf{e}_1) = T(1, 0) = (-1, 0)$ $T(\mathbf{e}_2) = T(0, 1) = (0, 1)$	$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$
Reflection about the line $y = x$ $T(x, y) = (y, x)$		$T(\mathbf{e}_1) = T(1, 0) = (0, 1)$ $T(\mathbf{e}_2) = T(0, 1) = (1, 0)$	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

In each case the **standard matrix** was obtained by finding the images of the standard basis vectors, i.e. $T(e_i)$, converting those images to column vectors, and then use those column vectors as successive columns of the standard matrix. I.e.

$$\mathbf{A} = [T(\mathbf{e}_1) \mid T(\mathbf{e}_2) \mid \cdots \mid T(\mathbf{e}_n)].$$

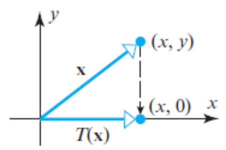
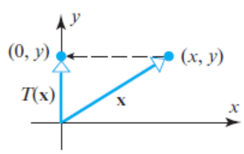
The following table shows the standard matrices for the reflections about the coordinate planes in $\mathbb{R}^3$.

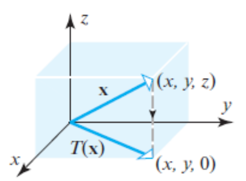
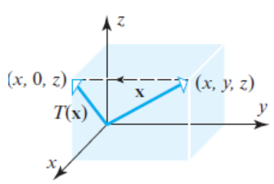
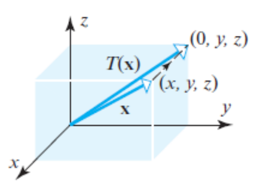
Operator	**Illustration**	**Images of $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$**	**Standard Matrix**
Reflection about the xy-plane $T(x, y, z) = (x, y, -z)$		$T(\mathbf{e}_1) = T(1, 0, 0) = (1, 0, 0)$ $T(\mathbf{e}_2) = T(0, 1, 0) = (0, 1, 0)$ $T(\mathbf{e}_3) = T(0, 0, 1) = (0, 0, -1)$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$
Reflection about the xz-plane $T(x, y, z) = (x, -y, z)$		$T(\mathbf{e}_1) = T(1, 0, 0) = (1, 0, 0)$ $T(\mathbf{e}_2) = T(0, 1, 0) = (0, -1, 0)$ $T(\mathbf{e}_3) = T(0, 0, 1) = (0, 0, 1)$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
Reflection about the yz-plane $T(x, y, z) = (-x, y, z)$		$T(\mathbf{e}_1) = T(1, 0, 0) = (-1, 0, 0)$ $T(\mathbf{e}_2) = T(0, 1, 0) = (0, 1, 0)$ $T(\mathbf{e}_3) = T(0, 0, 1) = (0, 0, 1)$	$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

2.1.2 Projection Operators

Matrix operators on R2 and R3 that map each point into its orthogonal projection onto a fixed line or plane through the origin are called projection operators (or more precisely, orthogonal projection operators).

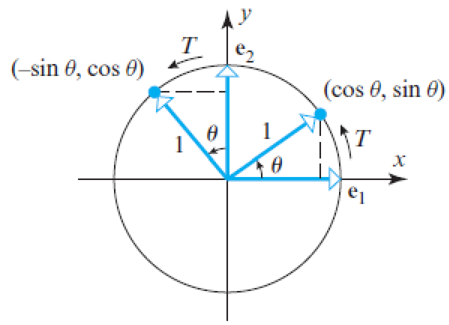
The following tables show the standard matrices for the **orthogonal projections** onto the coordinate axes in $\mathbb{R}^2$ and $\mathbb{R}^3$.

Operator	Illustration	Images of $\mathbf{e}_1$ and $\mathbf{e}_2$	Standard Matrix
Orthogonal projection onto the x-axis $T(x, y) = (x, 0)$		$T(\mathbf{e}_1) = T(1, 0) = (1, 0)$ $T(\mathbf{e}_2) = T(0, 1) = (0, 0)$	$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$
Orthogonal projection onto the y-axis $T(x, y) = (0, y)$		$T(\mathbf{e}_1) = T(1, 0) = (0, 0)$ $T(\mathbf{e}_2) = T(0, 1) = (0, 1)$	$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

Operator	Illustration	Images of $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$	Standard Matrix
Orthogonal projection onto the xy-plane $T(x, y, z) = (x, y, 0)$		$T(\mathbf{e}_1) = T(1, 0, 0) = (1, 0, 0)$ $T(\mathbf{e}_2) = T(0, 1, 0) = (0, 1, 0)$ $T(\mathbf{e}_3) = T(0, 0, 1) = (0, 0, 0)$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$
Orthogonal projection onto the xz-plane $T(x, y, z) = (x, 0, z)$		$T(\mathbf{e}_1) = T(1, 0, 0) = (1, 0, 0)$ $T(\mathbf{e}_2) = T(0, 1, 0) = (0, 0, 0)$ $T(\mathbf{e}_3) = T(0, 0, 1) = (0, 0, 1)$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
Orthogonal projection onto the yz-plane $T(x, y, z) = (0, y, z)$		$T(\mathbf{e}_1) = T(1, 0, 0) = (0, 0, 0)$ $T(\mathbf{e}_2) = T(0, 1, 0) = (0, 1, 0)$ $T(\mathbf{e}_3) = T(0, 0, 1) = (0, 0, 1)$	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

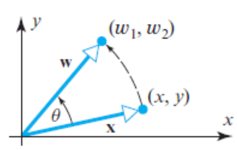
2.1.3 Rotation Operators

Matrix operators on $\mathbb{R}^2$ and $\mathbb{R}^3$ that move points along arcs of circles centered at the origin are called rotation operators. Let us consider how to find the standard matrix for the rotation operator $T : \mathbb{R}^2 \to \mathbb{R}^2$ that moves points counterclockwise about the origin through a positive angle θ.



The images of the standard basis vectors are
$T(\mathbf{e}_1) = T(1, 0) = (\cos(\theta), \sin(\theta))$ and $T(\mathbf{e}_2) = T(0, 1) = (-\sin(\theta), \cos(\theta))$.
So the standard matrix for T is

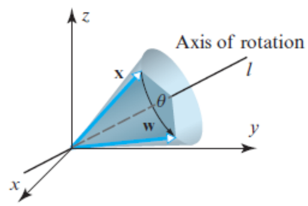
$$R_\theta = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

Operator	Illustration	Rotation Equations	Standard Matrix
Counterclockwise rotation about the origin through an angle θ		$w_1 = x\cos\theta - y\sin\theta$ $w_2 = x\sin\theta + y\cos\theta$	$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

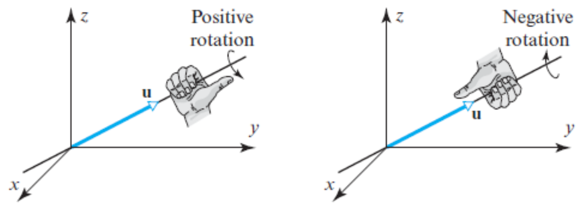
Example 2.2. Find the image of $x = (1, 1)$ under a rotation of $\pi/6$ radians about the origin.

Rotations in $\mathbb{R}^3$

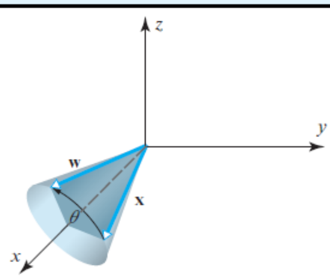
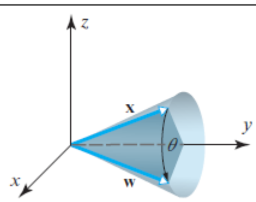
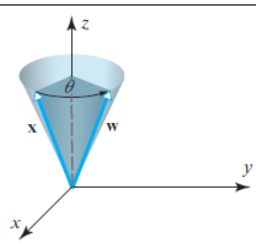
A rotation of vectors in $\mathbb{R}^3$ is commonly described in relation to a line through the origin called the axis of rotation and a unit vector **u** along that line. The unit vector and what is called the **right-hand rule** can be used to establish a sign for the angle of rotation by cupping the fingers of your right hand so they curl in the direction of rotation and observing the direction of your thumb. If your thumb points in the direction of **u**, then the angle of rotation is regarded to be positive relative to **u**, and if it points in the direction opposite to **u**, then it is regarded to be negative relative to **u**.



(a) Angle of rotation



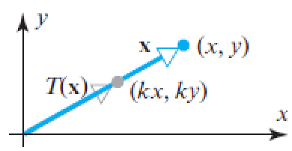
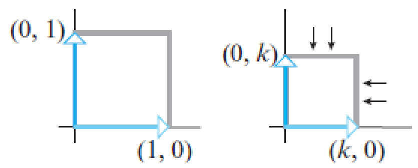
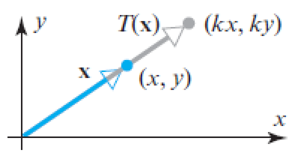
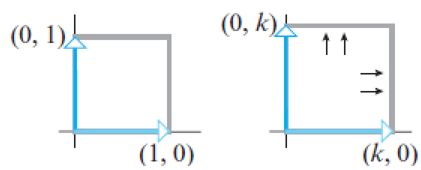
(b) Right-hand rule

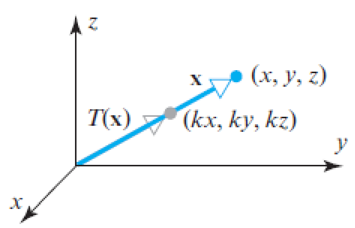
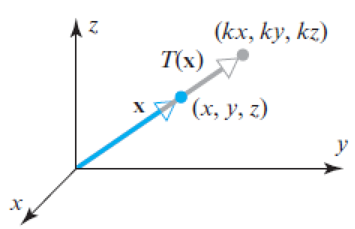
Operator	Illustration	Rotation Equations	Standard Matrix
Counterclockwise rotation about the positive x-axis through an angle θ		$w_1 = x$ $w_2 = y\cos\theta - z\sin\theta$ $w_3 = y\sin\theta + z\cos\theta$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$
Counterclockwise rotation about the positive y-axis through an angle θ		$w_1 = x\cos\theta + z\sin\theta$ $w_2 = y$ $w_3 = -x\sin\theta + z\cos\theta$	$\begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$
Counterclockwise rotation about the positive z-axis through an angle θ		$w_1 = x\cos\theta - y\sin\theta$ $w_2 = x\sin\theta + y\cos\theta$ $w_3 = z$	$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

2.1.4 Dilations and Contractions

Let k be a nonnegative scalar. The operator $T(\mathbf{x}) = kx$ on $\mathbb{R}^2$ or $\mathbb{R}^3$ has the effect of increasing or decreasing the length of each vector by a factor of k.
- If $0 \leq k < 1$ the operator is called a **contraction** with factor k.
- If $k > 1$, T is called a **dilation** with factor k.
- If $k = 1$, then T is the identity operator.

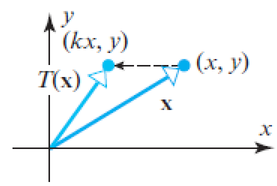
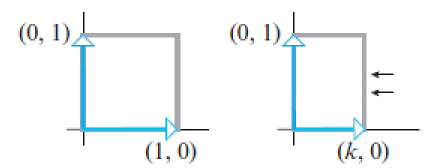
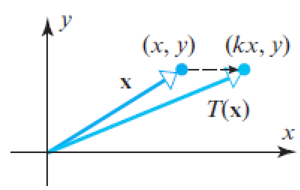
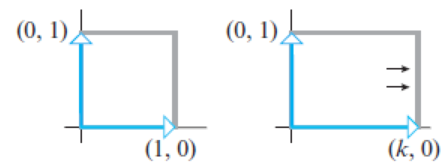
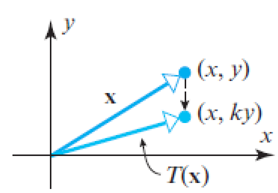
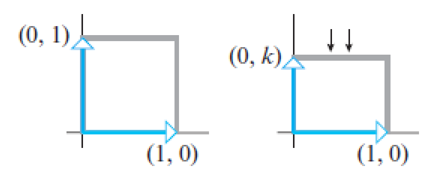
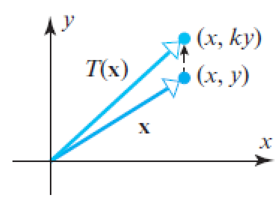
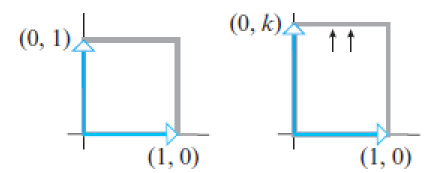
The following tables illustrate these operators.

Operator	Illustration $T(x, y) = (kx, ky)$	Effect on the Unit Square	Standard Matrix
Contraction with factor k in R^2 $(0 \leq k < 1)$			$\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$
Dilation with factor k in R^2 $(k > 1)$			

Operator	Illustration $T(x, y, z) = (kx, ky, kz)$	Standard Matrix
Contraction with factor k in R^3 $(0 \leq k < 1)$		$\begin{bmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{bmatrix}$
Dilation with factor k in R^3 $(k > 1)$		

2.1.5 Expansions and Compressions

In a dilation or contraction of $\mathbb{R}^2$ or $\mathbb{R}^3$, all coordinates are multiplied by a nonnegative factor k. If only one coordinate is multiplied by k, then, depending on the value of k, the resulting operator is called a **compression** or **expansion** with factor k in the direction of a coordinate axis.
This is illustrated in the following table for $\mathbb{R}^2$.

Operator	Illustration $T(x, y) = (kx, y)$	Effect on the Unit Square	Standard Matrix
Compression in the x-direction with factor k in R^2 $(0 \le k < 1)$			$\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$
Expansion in the x-direction with factor k in R^2 $(k > 1)$			
Operator	**Illustration $T(x, y) = (x, ky)$**	**Effect on the Unit Square**	**Standard Matrix**
Compression in the y-direction with factor k in R^2 $(0 \le k < 1)$			$\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$
Expansion in the y-direction with factor k in R^2 $(k > 1)$			

2.1.6 Shears

A matrix operator of the form

$$T(x, y) = (x + ky, y)$$

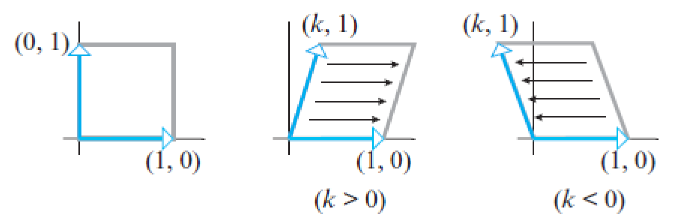
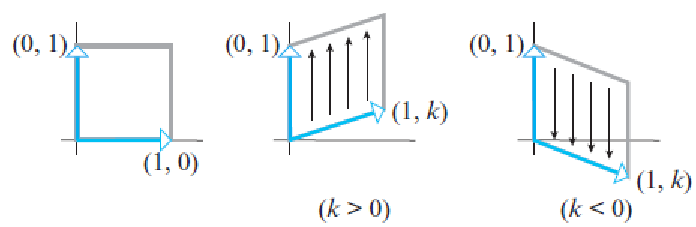
translates a point (x, y) in the xy-plane parallel to the x-axis by an amount ky that is proportional to the y-coordinate of the point. This operator leaves the points on the x-axis fixed, but as we progress away from the x-axis, the translation distance increases. We call this operator the **shear in the x-direction** by a factor k.

- Similarly, a matrix operator of the form

$$T(x, y) = (x, y + kx)$$

is called the **shear in the y-direction** by a factor k.

- The following table which illustrates the basic information about shears in $\mathbb{R}^2$, shows that a shear is in the positive direction if $k > 0$ and the negative direction if $k < 0$.

Operator	Effect on the Unit Square	Standard Matrix
Shear in the x-direction by a factor k in R^2 $T(x, y) = (x + ky, y)$		$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$
Shear in the y-direction by a factor k in R^2 $T(x, y) = (x, y + kx)$		$\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$

Example 2.3. Describe each of the following matrix operators for the given standard matrix, and show its effect on the unit square.

(a) $A_1 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ (b) $A_2 = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$ (c) $A_3 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ (d) $A_4 = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$

<u>**Solution:**</u>

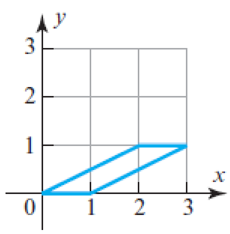
(a) The matrix $\mathbf{A}_1$ corresponds to a shear in the x-direction by a factor 2.

(b) The matrix $\mathbf{A}_2$ corresponds to a shear in the x-direction by a factor -2.

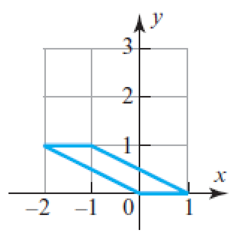
(c) The matrix $\mathbf{A}_3$ corresponds to a dilation with factor 2.

(d) The matrix $\mathbf{A}_4$ corresponds to an expansion in the x-direction with factor 2.

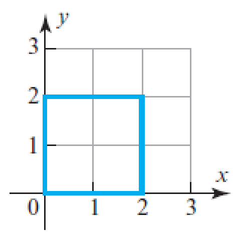
The effects of these operators on the unit square are shown below.



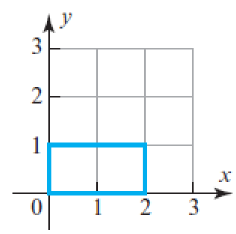
A_1



A_2



A_3



A_4