Relations & Functions III

TU152: Fundamental Mathematics

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TU

2/2013

Inverse of A Function Vs. Inverse Function

Inverse of A Function

Suppose $f : X \to Y$ is a function. Then we can write

$$y = f(x), x \in X \qquad \text{or} \qquad f = \{(x, y) | x \in X, y = Y\}.$$

The **inverse of function** f, denoted f^{-1}, is defined as

$$\boxed{f^{-1}(y) = x \quad \Leftrightarrow \quad y = f(x)} \qquad \text{or} \qquad \boxed{f^{-1} = \{(y, x) | (x, y) \in f\}}.$$

Note that the inverse of function f may or may not be a function. If f^{-1} is a function f^{-1} is called the **inverse function** of f.

Inverse Function

Suppose $f : X \to Y$ is a <u>*one-to-one correspondence or bijective (one-to-one and onto)*</u>.
Then the **inverse of function** f, f^{-1} , is a function and it is called the **inverse function of** f. As before, $f^{-1} : Y \to X$ is defined as:

$$\boxed{f^{-1}(y) = x \quad \Leftrightarrow \quad y = f(x)} \qquad \text{or} \qquad \boxed{f^{-1} = \{(y, x) | (x, y) \in f\}}.$$

Example: Let $X = \{1, 2, 3\}$ and $Y = \{-1, -2, -3\}$.
Let $f = \{(1, -3), (2, -1), (3, -3)\}$ be a function from X to Y.
Find the inverse of f, f^{-1} and determine whether it is the inverse function or not.

Example: Let $f = \{(x, y) \in \mathbb{R} \times \mathbb{R} | y = 2x - 3\}$ be a function. Find the inverse of the function f and determine whether it is the inverse function of f or not.

Definition (Composite Function)

Let $f : X \to \widehat{Y}$ and $g : Y \to Z$ be functions with the property that the range of f is a subset of the domain of g, $\widehat{Y} \subseteq Y$. Define a new function $g \circ f : X \to Z$ as follows:

$$(g \circ f)(x) = g(f(x)) \text{ for all } x \in X,$$

where

- $g \circ f$ is read "g circle f" and
- $g(f(x))$ is read "g of f of x."

The function $g \circ f$ is called the **composition** of f and g.

Remarks:

- Composition of functions is **not a commutative operation**.
  For general functions f and g, $f \circ g$ need not necessarily equal $g \circ f$ (although the two may be equal).

Definition

For $f : X \to \widehat{Y}$ and $g : Y \to Z$ with $\widehat{Y} \not\subseteq Y$, but $\widehat{Y} \cap Y \neq \emptyset$. We can still define a new function $g \circ f : \widehat{X} \to Z$ as follows:

$$(g \circ f)(x) = g(f(x)) \text{ for all } x \in \widehat{X} \subseteq X.$$

Notice that when $\widehat{Y} \subseteq Y$, the domain $\widehat{X}$ of $g \circ f$ will be the same as the domain X of f. However $\widehat{Y} \not\subseteq Y$, but $\widehat{Y} \cap Y \neq \emptyset$, $\widehat{X}$ may be smaller than X.

Example: Let $f : \mathbb{Z} \to \mathbb{Z}$ be the successor function and let $g : \mathbb{Z} \to \mathbb{Z}$ be the squaring function. Then $f(n) = n + 1$ for all $n \in \mathbb{Z}$ and $g(n) = n^2$ for all $n \in \mathbb{Z}$.

- Find the compositions $g \circ f$ and $f \circ g$.
- Is $g \circ f = f \circ g$? Explain.

Example: Let $X = \{1, 2, 3\}$, $\widehat{Y} = \{a, b, c, d\}$, $Y = \{a, b, c, d, e\}$, and $Z = \{x, y, z\}$.
Define functions $f : X \to \widehat{Y}$ and $g : Y \to Z$ by

- $f(1) = c$, $f(2) = b$, $f(3) = a$,
- $g(a) = y, g(b) = y, g(c) = z, g(d) = z, g(e) = z$.

Draw the arrow diagrams for f, g, and $g \circ f$.
Determine the range for each of f, g, and $g \circ f$.

Example: Let $f(x) = \frac{2}{\sqrt{x-1}}$ and $g(x) = x^2 + 3$. Find $g \circ f$ and $f \circ g$. Find $(g \circ f)(2)$.

Example (Composition with the Identity Function):
Let $X = \{a, b, c, d\}$ and $Y = \{u, v, w\}$, and suppose $f : X \to Y$ is given by

$$f(a) = u, \quad f(b) = v, \quad f(c) = v, \quad f(d) = u.$$

Let $I_X : X \to X$ and $I_Y : Y \to Y$ with $I_X(x) = x$ and $I_Y(y) = y$ be identity functions.
Find $f \circ I_X$ and $I_Y \circ f$.

Composition with an Identity Function

Theorem: Composition with an Identity Function

If f is a function from a set X to a set Y , and I_X is the identity function on X, and I_Y is the identity function on Y, i.e. ,

$$I_X(x) = x \quad \forall x \in X, \qquad I_Y(y) = y \quad \forall y \in Y$$

then

$$\boxed{f \circ I_X = f \qquad \text{and} \qquad I_Y \circ f = f.}$$

Let f be a function from a set X to a set Y, and suppose f has an inverse function f^{-1}. Recall that f^{-1} is the function from Y to X with the property that

$$f^{-1}(y) = x \quad \Leftrightarrow \quad f(x) = y.$$

What happens when f is composed with f^{-1}? Or when f^{-1} is composed with f ?

Example: Let $X = \{a, b, c\}$ and $Y = \{x, y, z\}$. Define $f : X \to Y$ by

$$f(a) = z, \quad f(b) = x, \quad f(c) = y.$$

1. Draw the arrow diagram for f.
2. Show that the inverse function f^{-1} exists for the function f.
3. Find f^{-1} and draw the arrow diagram for f^{-1}.
4. Find and draw the arrow diagrams for $f^{-1} \circ f$ and $f \circ f^{-1}$. Compare with I_X and I_Y.

Example (continued):

Composition of a Function with Its Inverse

Theorem:Composition of a Function with Its Inverse

If $f : X \to Y$ is a one-to-one and onto function with inverse function $f^{-1} : Y \to X$, then

$$f^{-1} \circ f = I_X \qquad \text{and} \qquad f \circ f^{-1} = I_Y.$$

Example: Show that the following statement is true.

Theorem: The composition of two injective(one-to-one) functions is also injective

If $f : X \to Y$ and $g : Y \to Z$ are both **one-to-one** functions, then $g \circ f$ is **one-to-one**.

Example: Show that the following statement is true.

Theorem: The composition of two surjective(onto) functions is also surjective(onto)

If $f : X \to Y$ and $g : Y \to Z$ are both **onto** functions, then $g \circ f$ is **onto**.