CHAPTER 10

Applications of Integration in Economics

Topics
- Total revenue function from marginal revenue function - Total cost function from marginal cost function - Profit function from MR-MC - Utility function from marginal utility function - Consumption and saving functions from marginal propensity functions - Consumer surplus, producer surplus and total surplus - First degree price discrimination

Total vs. Marginal

Total function $\xrightarrow{\text{Derivative}}$ Marginal f^n (rate of change of total function)

$\xleftarrow{\text{Anti - derivative}}$

Antiderivative or Integration

"Primitive f^n" $\xrightarrow{\text{differentiation}}$ "derived f^n / derivative f^n"

$F(x) + C$ $\quad \frac{d}{dx}(F(x) + C) \quad$ $f(x)$; C is a constant

$\frac{d}{dx} F(x) = f(x)$

know $\quad$ find

$y = x^2 + C$ $\quad \frac{dy}{dx} = ?$

$\frac{dy}{dx} = 2x$

On the contrary

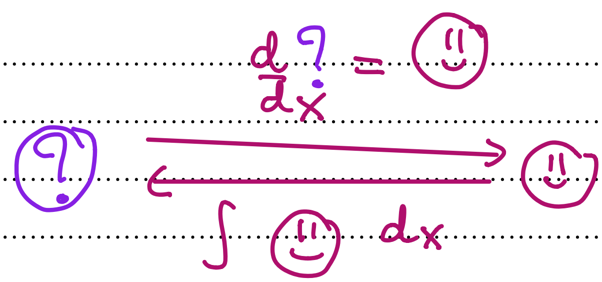
$F(x) + C \xleftarrow{\text{Integration}} f(x)$

find $\quad$ know

integral $\quad \int f(x)\, dx = F(x) + C \quad$ integrand

$x^2 + C \quad \int 2x\, dx \quad$ know $\frac{dy}{dx} = 2x$

TIP



Indefinite Integral

$$\boxed{\int f(x)dx = F(x) + C} \quad \text{with} \quad \frac{dF(x)}{dx} = f(x)$$

$$\frac{d}{dx}(F(x) + C) = f(x)$$

Basic Rule of Integration

The power rule

$$\int x^n dx = \frac{1}{n+1}x^{n+1} + c \quad (n \neq -1)$$

The exponential rule

$$\int e^x dx = e^x + c$$

The logarithmic rule

$$\int \frac{1}{x}dx = \ln x + c$$

$$\int f'(x)e^{f(x)}dx = e^{f(x)} + c$$

$$\int \frac{f'(x)}{f(x)}dx = \ln f(x) + c \quad [f(x) > 0]$$

<u>Rule of operation</u>

Integral of a sum

$$\int [f(x) + g(x)]dx = \int f(x)dx + \int g(x)dx$$

Integral of a multiple

$$\int kf(x)dx = k\int f(x)dx$$

<u>Rule Involving Substitution</u>

the substitution rule

$$\int g(x)dx = \int f(u)\frac{du}{dx}dx = \int f(u)du = F(u) + c \quad ; g(x) = f(u)\frac{du}{dx}$$

Integration by parts

$$\int v du = uv - \int u dv$$

Definite Integrals

From Indefinite Integral:

$$\int f(x)dx = F(x) + C$$

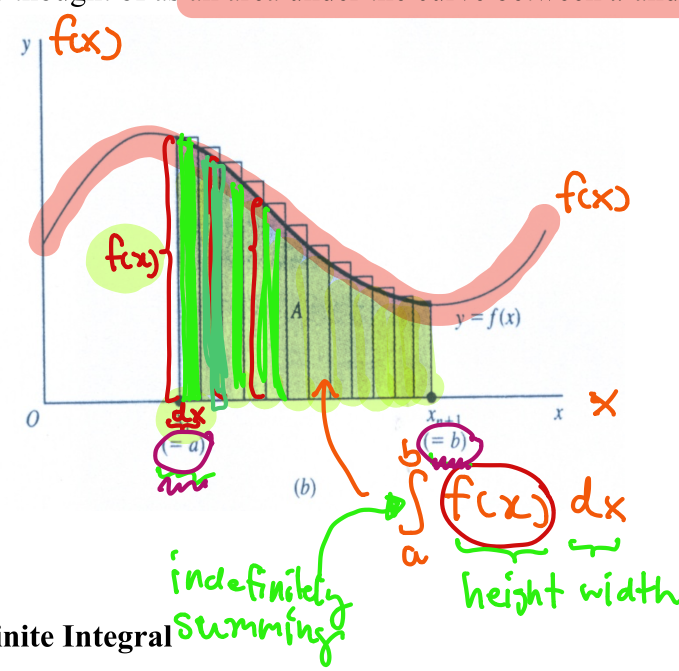
If we choose a and b , *where* $a < b$, in Domain of x *and find the difference:*

$$[F(b)+c]-[F(a)+c] = F(b)-F(a)$$

We call this "definite integral of f(x) from a to b" , with a be lower limit of integration and b the upper limit of integration.

$$\boxed{\int_a^b f(x)dx = F(x)\Big|_a^b = F(b)-F(a)}$$

A definite Integral can be thought of as an area under the curve between a and b.



Some Properties of Definite Integral

<u>Property I</u>: *The interchange of the limits of integration changes the sign of the definite integral:*

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

<u>Property II</u>: *A definite integral has a value of zero when the two limits of integration are identical*

$$\int_a^a f(x)dx = F(a)-F(a) = 0$$

Property III: *A definite integral can be expressed as a sum of a finite number of definite subintegrals as follows*

$$\int_a^d f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx + \int_c^d f(x)dx \quad (a < b < c < d)$$

Property IV:

$$\int_a^b -f(x)dx = -\int_a^b f(x)dx$$

Property V:

$$\int_a^b kf(x)dx = k\int_a^b f(x)dx$$

Property VI:

$$\int_a^b \left[f(x) \pm g(x)\right]dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)dx$$

Property VII: *(Integration by part)*

$$\int_{x=a}^{x=b} vdu = uv\Big|_{x=a}^{x=b} - \int_{x=a}^{x=b} udv$$

$$\frac{d}{dx}T(x) = M(x)$$

$$\int M(x)\,dx = F(x) + C = T(x) \quad ; \quad \frac{dF(x)}{dx} = M(x)$$

C can be figured out from initial or boundary condition

ex. $T(x=0) = 35$

$$\int_a^b f(x)\,dx = (F(b)+C) - (F(a)+C)$$

$$= F(b) - F(a)$$

$$= F(x)\Big|_a^b$$

= area below f(x) above x axis from x=a to x=b

Applications of Integration in Economics

Recovering Total function from Marginal function

Integrate MC to get TC

Let marginal cost function be MC(Q). We can find the underlying total cost as:

$$TC(Q) = \int MC(Q)dQ$$

Example: MC(Q) = 3 + 2Q

TC(Q) = $\int (3 + 2Q)\, dQ$

$= \int 3\, dQ + \int 2Q\, dQ$

$= (3Q + C) + \left(\frac{2Q^2}{2} + C\right) = \underbrace{3Q + Q^2}_{TVC} + \underbrace{C}_{TFC}$

Note: Q: What is the constant in a total cost function?

A: It is fixed costs, so using a marginal cost function we are unable to recover the level of fixed costs that a firm faces. To figure out the total fixed cost, we need initial condition or boundary condition.

if $TC(Q=0) = 50$ $\therefore$ $TC(Q=0) = C = 50 = TFC$

if $TC(Q=5) = 100$ $\therefore$ $TC(Q=5) = 15 + 25 + C = 100$

$C = 60 = TFC$

Integrate MR to get TR

$$TR(Q) = \int MR(Q)dQ$$

Integrate MP$_L$ to get Production function

$$TP_L(L) = \int MP_L(L)dL$$

Try: Find TP_L when $MP_L = \alpha$

Integrate MPS to get S

Let Mr. Yahoo's marginal propensity to save be $S'(Y) = 0.3 - 0.1Y^{-\frac{1}{2}}$. If Mr. Yahoo has US$ 81, he will not have any saving. What is Mr.Yahoo's saving function?

$$S(Y) = \int MPS(Y)\, dY$$

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Integrate MPC to get C

$$C(Y) = \int MPC(Y)\, dY$$

Try: Find $C(Y)$ when $MPC(Y) = b$

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Integrate Marginal Profit function to get Total Profit function

$$Marginal\ Profit(MP(Q)) = \frac{d\pi(Q)}{dQ} = \frac{d(TR(Q) - TC(Q))}{dQ} = MR(Q) - MC(Q)$$

$$\pi(Q) = \int MP(Q)\, dQ$$

If $MR = 50 - 2Q$ and $MC = 10 + Q$ and a firm produces at $\hat{q} = 10$, how much is the profit of this firm?

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Consumer and Producer Surplus

a competitive mkt.

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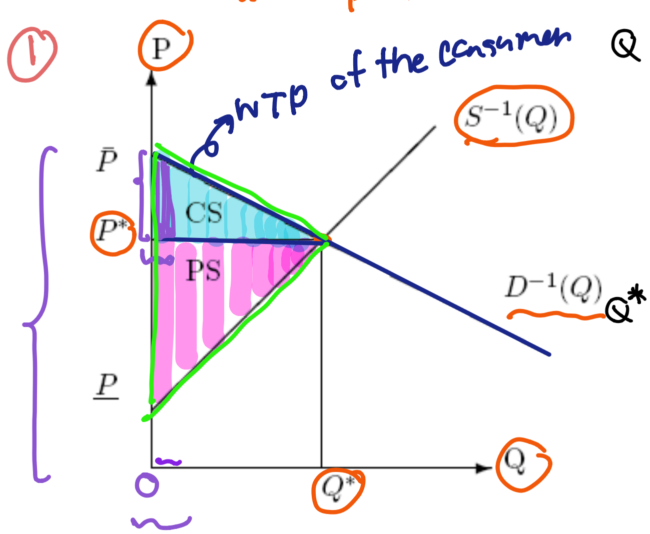

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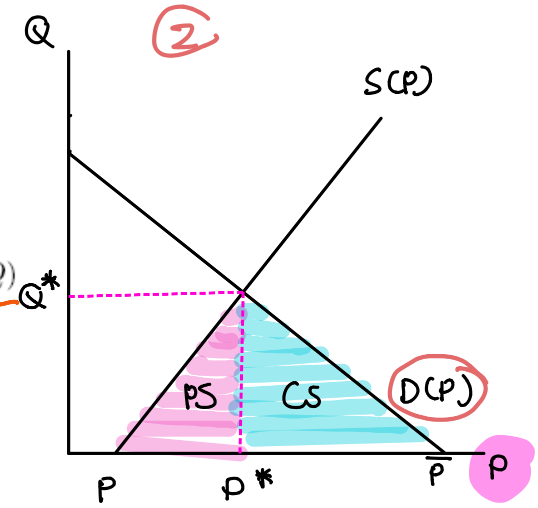

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CS = $\int_0^{Q^*} D^{-1}(Q)\,dQ - P^*Q^* = \int_0^{Q^*} \left(D^{-1}(Q) - P^*\right) dQ$

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CS = $\int_{P^*}^{\bar{P}} D(P)dP$

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PS = $P^*Q^* - \int_0^{Q^*} S^{-1}(Q)\,dQ = \int_0^{Q^*} \left(P^* - S^{-1}(Q)\right) dQ$

2

PS = $\int_{\underline{P}}^{P^*} S(P)dP$

TSW(Total Social Welfare) = CS + PS

$= \int_0^{Q^*} \left(D^{-1}(Q) - S^{-1}(Q)\right) dQ$

= the area above inverse supply, under inverse demand.

Example:

Let demand function be $Q^D = D(P) = \frac{25}{2} - \frac{1}{2}P$ $\quad \therefore P^D = D^{-1}(Q) = 25 - 2Q$

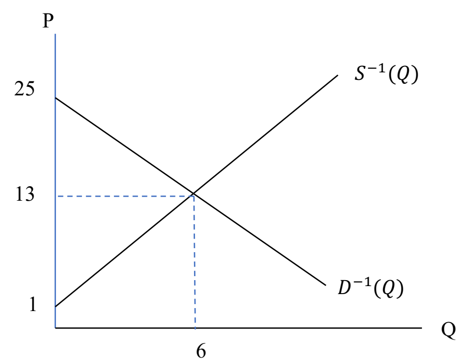
Let supply function be $Q^S = S(P) = -\frac{1}{2} + \frac{1}{2}P$ $\quad \therefore P^S = S^{-1}(Q) = 1 + 2Q$

Find consumer surplus and producer surplus

Competitive market

At equilibrium,

$$Q^D = Q^S$$
$$D(P) = S(P)$$
$$\frac{25}{2} - \frac{1}{2}P = -\frac{1}{2} + \frac{1}{2}P$$
$$P^* = 13, Q^* = 6$$



$CS = \int_0^6 (25 - 2Q)dQ - 13(6)$ $\qquad PS = 13(6) - \int_0^6 (1 + 2Q)dQ$

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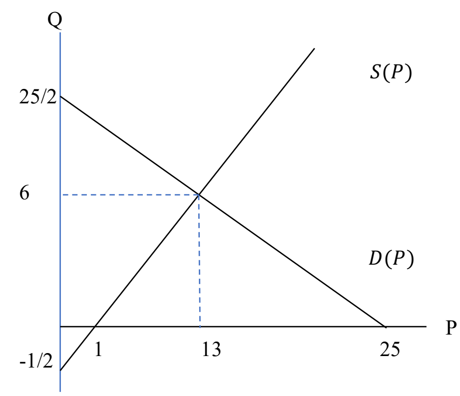

$$CS = \int_{13}^{25}\left(\frac{25}{2} - \frac{1}{2}P\right)dP \qquad PS = \int_{1}^{13}\left(-\frac{1}{2} + \frac{1}{2}P\right)dP$$

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$$TSW = CS + PS = \int_{0}^{6} [(25 - 2Q) - (1 + 2Q)]\, dQ$$

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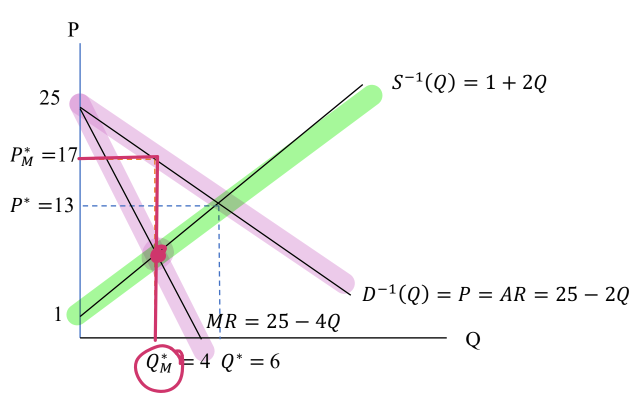
Monopoly market: Find dead weight loss



$CS_M = \int_0^{Q_M^*} AR(Q)dQ - P_M^* Q_M^*$ $\qquad$ $PS_M = P_M^* Q_M^* - \int_0^{Q_M^*} MC(Q)dQ$

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$TSW_M = CS_M + PS_M$

$$DWL = Total\ Social\ Welfare_{monopolist} - Total\ Social\ Welfare_{competitive}$$
$$= -\int_{Q_M^*}^{Q_C^*} [AR(Q) - MR(Q)]dQ$$

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First Degree Price Discrimination or Perfect Price Discrimination

Let market demand function of a monopolist be $P = 274 - Q^2$ and $MC = 4 + 3Q$

Find consumer surplus when:

(a.) the monopolist charges one price which maximizes the firm profit

(b.) the monopolist does perfect price discrimination. How much are total revenue and producer surplus when the monopolist does perfect price discrimination?

(a.) $\pi = TR - TC = (274 - Q^2)Q - \int (4 + 3Q)\,dQ$

$\pi = (274Q - Q^3) - (4Q + \frac{3Q^2}{2} + TFC)$

FOC: $\frac{d\pi}{dQ} = 0$: $MR - MC = 0$

$\underbrace{(274 - 3Q^2)}_{MR} - \underbrace{(4 + 3Q)}_{MC} = 0$

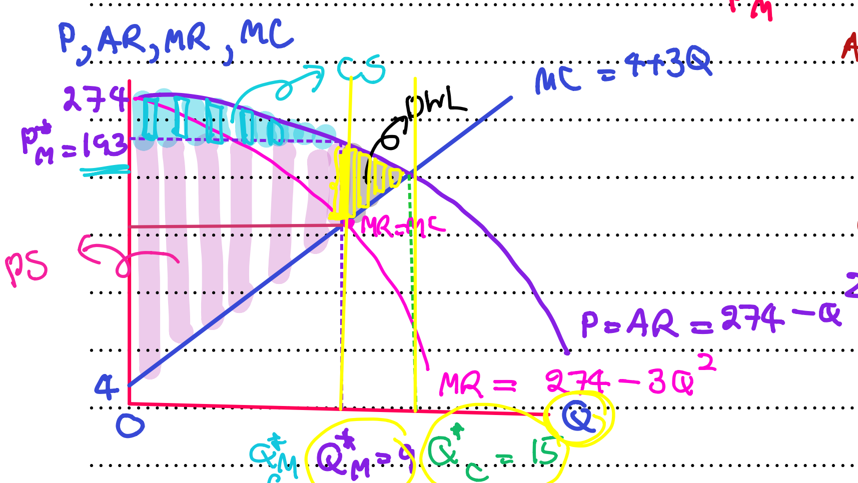
$270 - 3Q - 3Q^2 = 0$

$Q^2 + Q - 90 = 0$

$(Q + 10)(Q - 9) = 0$

$Q^*_M = \cancel{-10}, 9$

$P^*_M = 274 - 9^2 = 193$



At competitive mkt. equilibrium

$Q^D = Q^S$;

$274 - Q^2 = 4 + 3Q$; $P = MC$

$Q^2 + 3Q - 270 = 0$

$(Q + 18)(Q - 15) = 0$

$Q^*_C = 15, \cancel{-18}$

$CS = \int_0^9 (AR - P^*_M)\,dQ$

$= \int_0^9 (274 - Q^2 - 193)\,dQ$

$= \int_0^9 (81 - Q^2)\,dQ = \left[81Q - \frac{Q^3}{3}\right]_0^9 = \left(81(9) - \frac{9^3}{3}\right) - 0 = 486$

$$DWL = \text{Total Welfare}_M - \text{Total Welfare}_C$$

infinite sum $\sum$

$$= -\int_{Q_M^*}^{Q_C^*} (AR - MC)\, dQ$$

$$= -\int_{9}^{15} \left((274 - Q^2) - (4 + 3Q)\right) dQ$$

$$= -\int_{9}^{15} (270 - 3Q - Q^2)\, dQ$$

$$= \left[270Q - \frac{3Q^2}{2} - \frac{Q^3}{3}\right]_9^{15}$$

$$= -\left[270(15) - \frac{3(15)^2}{2} - \ldots\ldots\right]$$

$$= -522$$

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