EE 415/418 Game Theory

Lecture 14. Static Games of Incomplete Information

- 3.2 A
- Application – Mixed Strategy Revisited
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- Application – Double Auction

3.2.A – Mixed Strategy Revisited

- Remind that BNE is a set of strategies for each type of player that maximizes his expected value given other players' strategies.
- This is essentially the same definition as in NE except for the additional uncertainty about the type of the other players.
- When you play mixed NE, it implies your opponent is uncertain about your pure strategy, and know that your choice depends on your type. Thus, mixing your strategy implies incomplete information.

- Recall the game, Battle of Sexes

Pat

Chris	Opera	Fight
Opera	2,1	0,0
Fight	0,0	1,2

- There are two pure strategy NE's (Opera, Opera) and (Fight, Fight).
- And one mixed strategy, Chris plays Opera with probability 2/3 and Pat plays Fight with probability 2/3.

- Consider a slight modification of the game

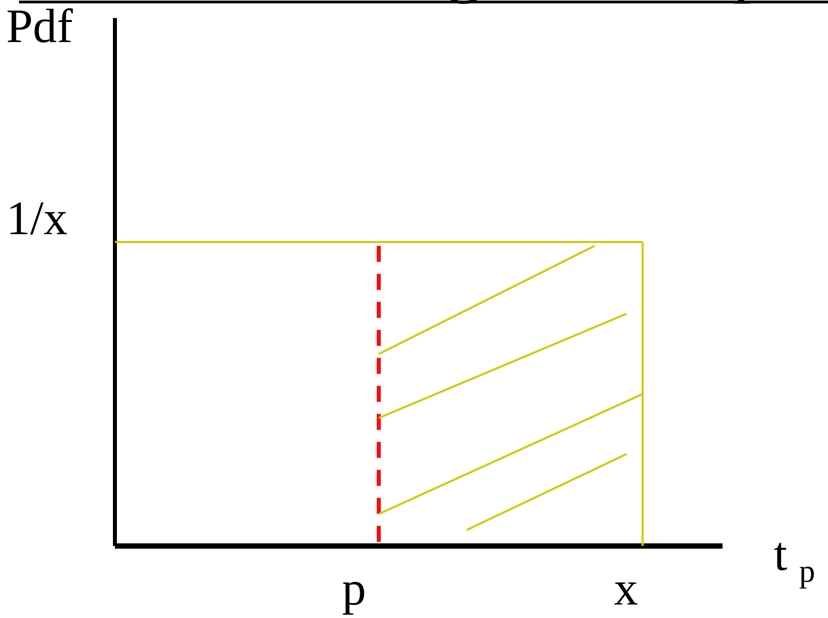
		Pat: Opera	Pat: Fight
Chris	Opera	$2+t_c$,1	0,0
Chris	Fight	0,0	1,$2+t_p$

- Players' payoffs depend on the type tc and tp.
- Both t_c and t_p are private information.
- Both t_c and t_p are randomly drawn from uniform distribution $(0, x)$.

- In this game, strategy spaces are $A_c = A\_p$ ={Opera, Fight}, types are $T_c = T_c = [0, x]$, and the beliefs are $p_c(t_p) = p_p(t_c) = 1/x$.
- Suppose there exist a pure strategy BNE, the Chris will play Opera if t_c exceed some critical value c. Likewise, Pat will play Fight if t_p exceed p.

- In this game, finding BNE means we want to find strategies of Pat and Chris that depending on his and her types. For certain values of her types, tc, Chris will play Opera and otherwise plays Fight. Thus, we need to find values of *c* and *p* to support such BNE.

- For Pat, Type ≥ *p* will love to play Fight.
- Shaded area shows that the chance that Pat will choose Fight = 1 – p/x or (x-p)/x



	Pat: Opera (p/x)	Pat: Fight (1- p/x)
Opera	$2+t_c$,1	0,0
Fight	0,0	1,$2+t_p$

- Given Pat's strategy, if Chris plays Opera, then her expected payoffs will be

$$\left(\frac{p}{x}\right)(2+t_c)+\left(1-\frac{p}{x}\right)(0)=\left(\frac{p}{x}\right)(2+t_c)$$

- and if she plays Fight, her expected payoff will be

$$\left(\frac{p}{x}\right)(0)+\left(1-\frac{p}{x}\right)(1)=1-\frac{p}{x}.$$

Thus, playing Opera is optimal if and only if

$$t_c \geq \frac{x}{p}-3 \equiv c.$$

- Likewise, the Pat's expected payoffs from playing Fight and Opera are

$$\left(\frac{c}{x}\right)(2+t_p)+\left(1-\frac{c}{x}\right)(0)=\left(\frac{c}{x}\right)(2+t_p),$$

$$\left(\frac{c}{x}\right)(0)+\left(1-\frac{c}{x}\right)(1)=1-\frac{c}{x}.$$

- Playing Fight is optimal if and only if

$$t_p \geq \boxed{\frac{x}{p}-3\equiv p.}$$

- Solving for c and p simultaneously yields $p = c$ and $p^2 + 3p - x = 0$.
- The probability that Chis plays Opera and the probability that Pat play Fight are given by

$$1 - \frac{-3 + \sqrt{9 + 4x}}{2x}.$$

- The probability approaches 2/3 as x approaches 0.
- As incomplete information disappears, the players' strategies in BNE in the game of incomplete information approaches their behavior in mixed strategy NE in the game of complete information.

Ie. no type when x is 0.

3.2.B – An Auction

- Consider the following first-price, sealed-bid auction. There are two bidders, labeled $i = 1,2$.
- Bidder i has a valuation v_i for the good - that is, if bidder i gets the good and pays the price p, then i's payoff is $v_i - p$.
- The two bidders' valuations are independently and uniformly distributed on [0,1].
- Bids are constrained to be nonnegative.
- The bidders simultaneously submit their bids.
- The higher bidder wins the good and pays the price she bid; the other bidder gets and pays nothing.
- In case of a tie, the winner is determined by a flip of a coin.
- The bidders are risk-neutral.

In this game, type of player is simply his valuation.

- In this game, player i's strategy space is $[0, \infty)$, type space is $T_i = [0,1]$, the payoff function is

$$u_i(b_1, b_2; v_1, v_2) = \begin{cases} v_i - b_i & \text{if } b_i > b_j \\ (v_i - b_i)/2 & \text{if } b_i = b_j \\ 0 & \text{if } b_i < b_j \end{cases}$$

- Recall that in static Bayesian game, a strategy is a function from type to action, $b_i(v_i)$. And that BNE strategies are mutual best response, $b_1(v_1)$ is a best response to $b_2(v_2)$ and vise versa

- Given the symmetric nature of this game, we look for an equilibrium where each player follows an identical strategy.
- First, we might guess that the function b(v) is strictly increasing; i.e.. Higher valuations lead to higher bids. (ex. Linear increasing function) Thus, we can let V(b) be its inverse function so that V(b) gives us the valuation of someone who bids b.
- If you bid b, your chance to win is when other's bid is less than b. This is exactly the probability that other player's valuation is less than V(b).

- Given the uniform distribution of v over [0 ,1], the probability that the other player's valuation is less than V(b) is equal to V(b).
- Hence, if you bid b when your valuation is v, your expected payoff is

$$(v - b).V(b) + 0.[1 - V(b)]. \qquad .....(*)$$

- (v - b) is your gain or consumer surplus if win, and 0 is your zero surplus if not win.
- Thus, your optimal bid, b, must maximize (*).
- FOC: $(v - b)V'(b) - V(b) = 0.$

- For each value of v, this FOC gives you b(v) as optimal.

- By the hypothesis of V(b) as an optimal bid, we can replace v with V(b):
- $(V(b) - b)V'(b) - V(b) = 0,$
- This is a first-order differential equation, and the solution is
- $V(b) = b + \sqrt{b^2 + 2C}$ where C is constant from integration.
- We need one information to determine C. Note that when v = 0, your optimal bid b = 0.

- $V(b) = b + \sqrt{b^2 + 2C}$ where C is constant from integration.

- when v = 0, your optimal bid b = 0 or V(b) = 0.
- Use this we get $0 = 0 + \sqrt{2C}$. Thus C = 0.
- Hence, V(b) = b + b = 2b. Thus, b = v /2.
- BNE is each player to bid half of valuation.
- The resulting unique and linear bid function is specific to the given distribution of valuation or types of players. In general, there can be many other equilibria in a more general game.

- Thus, in BNE, the pair $(b_1^*(v_1), b_2^*(v_2))$ solves

$$\max_{b_i}(v_i - b_i)\text{Prob}\{b_i > b_j^*(v_j)\} + \frac{1}{2}(v_i - b_i)\text{Prob}\{b_i = b_j^*(v_j)\}.$$

- Consider linear equilibrium
  $b_1^*(v_1) = a_1^* + c_1^* v_1$ and
  $b_2^*(v_2) = a_2^* + c_2^* v_2$.
- Suppose player j adopt $b_j(v_j) = a_j + c_j v_j$,
  then player i solves

$$\max_{b_i}(v_i - b_i)\text{Prob}\{b_i > a_j^* + c_j^* v_j\}.$$

Using the fact that you will not bid below j's minimum bid, a_j, and will not bid above j's maximum bid $a_j + c_j$, thus $\text{prob}(b_i = b_j(v_j)) = 0$. (last term in your equation)

- Since $v_j \sim U(0,1)$, thus

$$\text{Prob}\left(b_i > a_j^* + c_j^* v_j\right) = \text{Prob}\left(v_j < \frac{b_i - a_j^*}{c_j^*}\right)$$

$$= \frac{b_i - a_j^*}{c_j^*}.$$

- Player i's best response is therefore

$$b_i^*(v_i) = \begin{cases} (v_i + a_j^*)/2 & \text{if } v_i \geq a_j^* \\ a_j^* & \text{if } v_i < a_j^*. \end{cases}$$

- If $a_j^* \in (0,1)$, then the best response in previous equation is non-linear.
- So, in order to maintain our linear assumption, we must have $a_j^* \geq 1$ or $a_j^* \leq 0$.
- But the former case is not possible since it is optimal for the higher type to bid at least as much as the lower type optimal bid.
- Thus we have $a_j^* \leq 0$, giving $b_i^*(v_i) = (v_i + a_j^*)/2$, so that $a_i^* = a_j^*/2$ and $c_i^* = 1/2$.
- In BNE, $a_i^* = a_j^*/2$ and $a_i^* = a_j^*/2$, thus $a_i^* = 0$ for all i.
- The equilibrium strategy is $b_i^*(v_i) = v_i/2$.