Chapter 5 Inequalities Involving the Absolute Values

In this chapter we consider the inequalities involving the absolute values. Before anything else, we need to understand what the absolute value refers to.

Definition 5.1: For a real number *x*, the ***absolute value*** of *x*, denoted by $|x|$, is defined to be

$$|x| = \begin{cases} x & , x \geq 0 \\ -x & , x < 0 \end{cases}$$

Note: From the definition 5.1 we see that the absolute value of any real number x is greater than or equal to zero. Do not let the minus sign in the second line of the definition confuse you. In the case of $x < 0$ observe that $|x| = -x$ is a positive value. For instance, if $x = -2$ then $|-2| = -(-2) = 2$.

Section 5.1: Theorems about Inequalities Involving the Absolute Values

There are several important theorems which we can use to solve the inequalities involving the absolute values. We first introduce the theorems involving absolute values.

Theorem 5.1.1: For any real numbers *x*:

1) If $x \neq 0$ then $|x| > 0$
2) $|x| = 0$ if and only if $x = 0$

Proof:

Theorem 5.1.2: For any real numbers *x*:

1) $|-x| = |x|$

2) $|x| \le |x|$

Proof:

Theorem 5.1.3: For any real numbers *x*, and *a* is a positive real number:

1) $|x| < a$ if and only if $-a < x < a$.

2) $|x| \le a$ if and only if $-a \le x \le a$.

Proof:

Theorem 5.1.4: For any real numbers *x*, and *a* is a positive real number:

1) $|x| > a$ if and only if $x > a$ or $x < -a$.
2) $|x| \geq a$ if and only if $x \geq a$ or $x \leq -a$.

Proof:

Theorem 5.1.5: If *x* is a real number then $-|x| \leq x \leq |x|$.

Proof:

Theorem 5.1.6: For any real numbers x and y:

1) $|x+y| \le |x|+|y|$
2) $|x-y| \le |x|+|y|$
3) $\big||x|-|y|\big| \le |x-y|$

Proof:

Theorem 5.1.7: For any real numbers x and y:

1) $|x\,y| = |x||y|$
2) If $y \neq 0$ then $\left|\frac{1}{y}\right| = \frac{1}{|y|}$
3) If $y \neq 0$ then $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$

Proof:

Theorem 5.1.8: If x is a real number then $|x|^2 = x^2$.

Proof:

Theorem 5.1.9: For any real numbers x and y:

1) $|x| < |y|$ if and only if $x^2 < y^2$.
2) $|x| \le |y|$ if and only if $x^2 \le y^2$.

Proof:

Example 5.1.1: Let *x* be any real numbers, prove that $\left|x^2-1\right|<4\left|x+1\right|$.

Example 5.1.2: Let *x* and *y* be any real numbers, prove that if $\left|x+1\right|<\left|x+y\right|$ then $2x+1<2xy+y^2$.

Example 5.1.3: Let x be any real number where $2 \le x \le 3$, prove that $3 \le \left|x^2 - 2x + 3\right| \le 6$.

Example 5.1.4: Let x and y be any real numbers where $|x| < 2$, prove that if $y > 0$ then $|x + y| < y + 2$.

Example 5.1.5: Let x and y be any real numbers, prove that $\dfrac{|x+y|^2 - 6xy}{3} \le |x|^2 + |y|^2$.

5.2 Solving Inequalities Involving the Absolute Values

Solving inequality involving the absolute values we need to eliminate the absolute values first then use the method of solving the inequalities without the absolute values presented in the previous chapter.

Example 5.2.1: Find the solution set of the inequality $|x+2| \le 4$.

Example 5.2.2: Solve the inequality $2 < |2x - 1| \le 5$.

Example 5.2.3: Find the solution set of the inequality $|x+5|-|x| \geq 3$.

Example 5.2.4: Find the solution set of the inequality $\left|x^2+x-6\right|<6$.

Example 5.2.5: Solve the inequality $|x+2| \le |2x+3|$.

Example 5.2.6: Find the solution set of the inequality $\left|\frac{8}{x}\right| \geq x^2$.

Example 5.2.7: Solve the inequality $|x+1| > \dfrac{x^2}{|x|-1}$.

Example 5.2.8: Solve the inequality $\frac{1}{|x+2|}-\frac{x}{|x+1|}<1$.

Example 5.2.9: Solve the inequality $\left|x^2 - |x-1|\right| \geq 1$.

Example 5.2.10: Find the solution set of the inequality $\left|x+\left|x+2\right|\right|<3$.