1. Two individuals agree at date 0 to a forward contract that matures at date 2. The contract is written on an underlying asset that pays a dividend at date 1 equal to D_1. Let f_2 be the date 2 random payoff (profit) to the individual who is the long party in the forward contract. Also let m_{0i} be the stochastic discount factor over the period from dates 0 to i where $i = 1, 2$, and let $E_0[\cdot]$ be the expectations operator at date 0. What is the value of $E_0[m_{02}f_2]$? Explain your answer.

Let, S_i = price of underlying asset at date i

D_0 = dividend at date 0 that pay btw dates 0 and 1

F_{02} = forward price

$S_0 = E_0[m_{01}D_1] + E_0[m_{02}S_2]$

$= D_0 + E_0[m_{02}S_2]$

payoff to long party : $f_2 = S_2 - F_{02}$

Long forward position represents ownership in share of the underlying assets

1. short positioning (selling) underlying asset's dividends

2. borrowing same amount as repayment at date 2 ; F_{02}

$\therefore$ Using SDF approach to pricing,

$E[m_{02}f_2] = E_0[m_{02}(S_2 - F_{02})]$

$= E[m_{02}S_2] - E[m_{02}F_{02}]$; $E[m_{02}F_{02}] = E[m_{02}]F_{02}$

$= S_0 - D_0 - R_F^{-2}F_{02}$ $\qquad = R_F^{-2}F_{02}$

Then, in absence of arbitrage

$F_{02} = R_f^2(S_0 - D_0)$

$\therefore E[m_{02}f_2] = 0$ *

2. Assume that there is an economy populated by infinitely-lived representative individuals who maximize the lifetime utility function

$$E_0\left[\sum_{t=0}^{\infty} -\delta^t e^{-ac_t}\right]$$

where c_t is consumption at date t and $a > 0$, $0 < \delta < 1$. The economy is a Lucas (1978) endowment economy having multiple risky assets paying date t dividends that total d_t per capita. Write down an expression for the equilibrium per capita price of the market portfolio in terms of the assets' future dividends.

Lucas (1978) Price of risky asset

$$P_{it} = E_t\left[\sum_{j=1}^{T} \delta^j \frac{u_c(C_{t+j}^*)}{u_c(C_t^*)} d_{i,t+j}\right]$$

$$P_0 = E_0\left[\sum_{t=1}^{\infty} \frac{U_c(C_t,t)}{U_c(C_0,0)} d_t\right]$$; $U(C_t,t) = -\delta^t e^{-aC_t}$

$U_c(C_t,t) = a\delta^t e^{-aC_t}$

Since endowment economy with 1 share per individual ; $C_t = d_t$

So, $$P_0 = E_0\left[\sum_{t=1}^{\infty} \frac{U_c(C_t,t)}{U_c(C_0,0)} d_t\right] = E_0\left[\sum_{t=1}^{\infty} \delta^t e^{-a(d_t - d_0)} d_t\right]$$

3. For the Lucas model with labor income, show that assumptions (6.25) and (6.26) lead to the pricing relationship (6.27) and (6.28).

$C_t^* = d_t + y_t$ aggregate consumption depend on 1. dividend
2. labor income

$$P_{it} = E_t\left[\sum_{j=1}^{T} \delta^j \frac{u_c(C_{t+j}^*)}{u_c(C_t^*)} d_{t+j}\right]$$

$$= E_t\left[\sum_{j=1}^{\infty} \delta^j \left(\frac{C_{t+j}^*}{C_t^*}\right)^{\gamma-1} d_{t+j}\right]$$

$$= E_t\left[\sum_{j=1}^{\infty} \delta^j e^{(\gamma-1)\ln\left(\frac{C_{t+j}^*}{C_j^*}\right) + \ln\left(\frac{d_{t+j}}{d_t}\right)}\right]$$

Streming Consumption : $\ln\left(\frac{C_{t+j}^*}{C_t^*}\right) = j \cdot \mu_c + \sigma_c \sum_{i=1}^{j} \eta_{t+j}$

Streming Dividend $\ln\left(\frac{d_{t+j}}{d_t}\right) = j \cdot \mu_d + \sigma_d \sum_{i=1}^{j} \varepsilon_{t+j}$

$\Rightarrow \therefore \frac{C_{t+j}}{C_t} = e^{j\mu_c + \sigma_c + \sum_{i=1}^{j} \eta_{t+j}}$

We know that, $$\frac{P_t}{d_t} = E_t\left[\sum_{j=1}^{\infty} \delta^j \left(\frac{C_{t+j}^*}{C_t^*}\right)^{\gamma-1} \frac{d_{t+j}}{d_t}\right]$$

$$\frac{P_t}{d_t} = E_t\left[\sum_{j=1}^{\infty} \delta^j e^{(\gamma-1)[j\mu_c + \sigma_c \sum_{i=1}^{j} \eta_{t+j}] + [j\mu_d + \sigma_d \sum_{i=1}^{j} \varepsilon_{t+j}]}\right]$$

$$\frac{P_t}{d_t} = E_t\left[\sum_{j=1}^{\infty} \delta^j e^{[(\gamma-1)\mu_c + \mu_d] + \sum_{i=1}^{j} [(\gamma-1)\sigma_c \eta_{t+j} + \sigma_d \varepsilon_{t+j}]}\right]$$

Take expectation inside,

$$\frac{P_t}{d_t} = \sum_{j=1}^{\infty} \delta^j e^{(\gamma-1)j\mu_c + \frac{1}{2}(\gamma-1)^2\sigma_c^2 j + j\mu_d + \frac{1}{2}j\sigma_d^2 + \frac{1}{\cancel{2}}\cancel{2}(\gamma-1)\cdot\sigma_c\sigma_d\rho j}$$

$$\frac{P_t}{d_t} = \sum_{j=1}^{\infty} e^{[\ln\delta - (1-\gamma)\mu_c + \mu_d + \frac{1}{2}((1-\gamma)^2\sigma_c^2 + \sigma_d^2) - (1-\gamma)\sigma_c\sigma_d\rho]}$$

So,
$$= \frac{1}{1 - \delta e^{-(1-\gamma)\mu_c + \mu_d + \frac{1}{2}[(1-\gamma)^2\sigma_c^2 + \sigma_d^2] - (1-\gamma)\sigma_c\sigma_d\rho}} - 1$$

$$P_t = d_t \frac{\delta e^{-(1-\gamma)\mu_c + \mu_d + \frac{1}{2}[(1-\gamma)^2\sigma_c^2 + \sigma_d^2] - (1-\gamma)\sigma_c\sigma_d\rho}}{1 - \delta e^{-(1-\gamma)\mu_c + \mu_d + \frac{1}{2}[(1-\gamma)^2\sigma_c^2 + \sigma_d^2] - (1-\gamma)\sigma_c\sigma_d\rho}} \quad *$$

or

$$P_t = d_t \frac{\delta e^{\alpha}}{1 - \delta e^{\alpha}} \;;\; \alpha = -(1-\gamma)\mu_c + \mu_d + \frac{1}{2}[(1-\gamma)^2\sigma_c^2 + \sigma_d^2] - (1-\gamma)\sigma_c\sigma_d\rho$$

4. Consider a special case of the model of rational speculative bubbles discussed in this chapter. Assume that infinitely-lived investors are risk-neutral and that there is an asset paying a constant, one-period risk-free return of $R_f = \delta^{-1} > 1$. There is also an infinitely-lived risky asset with price p_t at date t. The risky asset is assumed to pay a dividend of d_t which is declared at date t and paid at the end of the period, date $t+1$. Consider the price $p_t = f_t + b_t$ where

$$f_t = \sum_{i=0}^{\infty} \frac{E_t\left[d_{t+i}\right]}{R_f{}^{i+1}} \; (1) \tag{1}$$

and

$$b_{t+1} = \begin{cases} \frac{R_f}{q_t} b_t + e_{t+1} & \text{with probability } q_t \\ z_{t+1} & \text{with probability } 1 - q_t \end{cases} \tag{2}$$

where $E_t\left[e_{t+1}\right] = E_t\left[z_{t+1}\right] = 0$ and where q_t is a random variable as of date $t-1$ but realized at date t and is uniformly distributed between 0 and 1.

4.a Show whether or not $p_t = f_t + b_t$ subject to the specifications in (1) and (2) is a valid solution for the price of the risky asset.

$$P_t = \sum_{i=0}^{\infty} \frac{E_t[d_{t+i}]}{R_f^{i+1}} + b_t$$

$$E_t[b_{t+1}] = \frac{R_f}{\cancel{q_t}} b_t \cancel{q_t} + \underbrace{E_t[e_{t+1}]}_{0} q_t + \underbrace{E_t[z_{t+1}]}_{0}(1-q_t) \; ; \; E_t[e_{t+1}] = E[z_{t+1}] = 0$$

$\therefore$ $E_t[b_{t+1}] = R_f b_t$ * valid solution for the price of the risky asset

(2) is satisfies $E_t[b_{t+1}] = R_f b_t$

4.b Suppose that p_t is the price of a barrel of oil. If $p_t \geq p_{solar}$, then solar energy, which is in perfectly elastic supply, becomes an economically efficient perfect substitute for oil. Can a rational speculative bubble exist for the price of oil? Explain why or why not.

∞

$$E_t[b_{t+1}] = R_f b_t \Rightarrow \text{So,} \lim_{i \to \infty} E[b_{t+i}] = \begin{cases} \infty & \text{if } b_t > 0 \\ -\infty & \text{if } b_t < 0 \end{cases}$$

Since oil is limited liability assets ; we cannot have a bubble path with price becoming negative.

So, must $b_t > 0 \rightarrow$ that $E[b_{t+i}] = \infty$; bubble component expected to increase infinitely.

But, $P_t \geq P_{solar}$; b_t cannot rise above $P_{solar} - P_t^*$, so this rational expectation cannot occur cause there is an upper bound on price which perfect substitute in perfectly elastic supply.

∴ bubble path that expected to increase infinitely do not exist.
b_t

4.c Suppose p_t is the price of a bond that matures at date $T < \infty$. In this context, the d_t for $t \leq T$ denotes the bond's coupon and principal payments. Can a rational speculative bubble exist for the price of this bond? Explain why or why not.

Since at date T, $P_t = f_t = d_t \rightarrow$ maturity After date T, $P_t = 0$

Its price cannot rationally be expected to satisfy equation and increase infinitely,

so rational speculative bubble do not exist for the price of this bond ∴ rational price : $P_t = P_t^*$

5. Consider an endowment economy with representative agents who maximize the following objective function:

$$\max_{C_s, \{\omega_{is}\}, \forall s, i} E_t \left[\sum_{s=t}^{T} \delta^s u(C_s) \right]$$

where $T < \infty$. Explain why a rational speculative asset price bubble could not exist in such an economy.

$$\max_{C_s, \{W_{is}\}, \forall s, i} E_t \left[\sum_{s=t}^{T} \delta^s u(C_s) \right] \; ; \; T < \infty$$

Imply that economy having finite horizon that asset prices couldn't be $p_t = f_t + b_t$ with $b_t \neq 0$

so at date T, $p_t = f_t = d_t$ as asset's final dividend payment.

bubble process : $E[b_{t+1}] = \delta^{-1} b_t$; since $b_T = 0$

so, $E_{T-1}[b_T] = E_{T-1}[0] = \delta^{-1} b_{T-1}$

$\therefore b_{T-1} = 0 \rightarrow$ implies $b_t = 0$ for all previous dates, $t < T-1$

bubble do not exist