Note Title 9/14/2016

Note on Complex Roots

$$y''(t) + a_1 y'(t) + a_2 y(t) = b$$

$$y_c = e^{ht}(A_1 e^{vit} + A_2 e^{-vit}) \quad \text{— } (\star)$$

$$y_c = e^{ht}[\underbrace{A_1(\cos vt + i\sin vt)}_{①} + \underbrace{A_2(\cos vt - i\sin vt)}_{②}] \quad \text{— } (\star\star)$$

$$\boxed{y_c = e^{ht}[A_5 \cos vt + A_6 \sin vt]} \quad \text{— } (\star\star\star)$$

where $A_5 = A_1 + A_2$ and $A_6 = (A_1 - A_2) i$

Let $A_1 = m + ni$

$A_2 = m - ni$

$\therefore A_5 = A_1 + A_2$

$= (m + ni) + (m - ni)$

$= \boxed{2m}$

$A_6 = (A_1 - A_2) i$

$= [(\cancel{m} + ni) - (\cancel{m} - ni)] i$

$= (2ni) i$

$= 2ni^2$

$= 2n(-1) \quad = \boxed{-2n}$

General Sol^n

$$y(t) = e^{ht}(A_5 \cos vt + A_6 \sin vt) + y_p$$

a) $h > 0$

$\underbrace{e^{ht}}$ will increase as $t \to \infty$

$\downarrow$

magnify $(A_5 \cos vt + A_6 \sin vt)$

The time path of $y(t)$ is explosive fluctuation around y_p

b) $h = 0$

$e^{ht} = 1 \Rightarrow (A_5 \cos vt + A_6 \sin vt)$ has a constant amplitude

The time path is uniform fluctuation

c) $h < 0$

$e^{ht} \to 0$ as $t \to \infty$

The time path is "damped" fluctuations

$$y(t) \to y_p \quad \text{as} \quad t \to \infty$$

In sum, the equilibrium is dynamically STABLE iff the time path is convergent.

- **Stability for second-order (linear) DEs**

$$y''(t) + a_1 y'(t) + a_2 y(t) = b \qquad — (1)$$

$$y(t) = A_1 e^{r_1 t} + A_2 e^{r_2 t} + y_p \qquad — (i)$$

$$y(t) = A_3 e^{rt} + A_4 t e^{rt} + y_p \qquad — (ii)$$

$$y(t) = e^{ht}(A_5 \cos vt + A_6 \sin vt) + y_p \qquad — (iii)$$

Eq(1) is called (globally) stable iff y_c tends to zero as $t \to \infty$ for all values of $A_1, \ldots, A_6$

(i) $r_1 < 0$ AND $r_2 < 0$

(ii) $r < 0$

(iii) $\underline{\underline{h}} < 0$

" real part is negative "

ⓒ Application : Inflation and Unemployment Rates

The Phillips Relation

$$w = f(U) \quad — (1)$$

$w = \dfrac{\dot{W}}{W}$ = the growth rate of nominal wage

U = the unemployment rate

$$p = \frac{\dot{P}}{P}$$

p = the growth rate of the price level

= the inflation rate

$$p = w - T \quad — (2)$$

$w - p = T$

T = labor productivity. (exogenous)

Assume $f(U) = \alpha - \beta U$; $\alpha, \beta > 0$

(1), (2) and $f(U)$

$$p = (\alpha - T) - \beta U \quad — (3)$$

Prof. Friedman

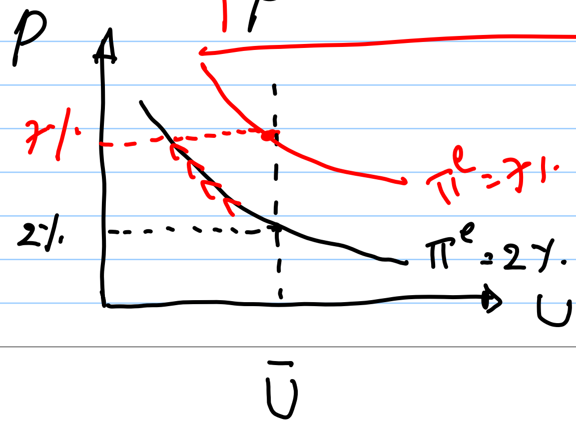
Expectations-Augmented Phillips Curve (Relation)

$$w = f(U) + g\pi^e \;;\; 0 < g \leq 1 \qquad \text{— (1)}'$$

π^e = the expected rate of inflation

$(1)', (2), f(U)$

$$\boxed{p = (\alpha - T) - \beta U + g\pi^e \qquad \text{— (4)}}$$



p, π^e, U

$$\boxed{\frac{d\pi^e}{dt} = j(p - \pi^e) \qquad \text{— (5)} \qquad 0 < j \leq 1}$$

$p > \pi^e \Rightarrow \pi^e$ is revised upward $\Rightarrow \frac{d\pi^e}{dt} > 0$

$p < \pi^e \Rightarrow \pi^e$ is revised downward $\Rightarrow \frac{d\pi^e}{dt} < 0$

(5) ⇒ "adaptive expectations" hypothesis

we introduce the feedback of MP

$$\boxed{\frac{dU}{dt} = -k(m - p) \qquad k > 0 \qquad \text{— (6)}}$$

$m = \frac{\dot{M}}{M}$ = the growth rate of (nominal) money supply.

$m - p$ = the growth rate of real balance.

M is exogenously determined.

(6) → $\frac{dU}{dt}$ is negatively related to the growth rate of real balance.

Find $\pi^e(t)$, $p(t)$, $U(t)$

✓ ✓

(4) → (5)

$$\frac{d\pi^e}{dt} = j(\alpha - \bar{\iota}) - j\beta U - j(1-g)\pi^e \qquad -(7)$$

Diff (7) w.r.t t

(6)

$$\frac{d^2\pi^e}{dt^2} = -j\beta\frac{dU}{dt} - j(1-g)\frac{d\pi^e}{dt} \qquad -(8)$$

sub (6) for $\frac{dU}{dt}$

$$\frac{d^2\pi^e}{dt^2} = -j\beta[-k(m-p)] - j(1-g)\frac{d\pi^e}{dt}$$

$$= j\beta km - j\beta kp - j(1-g)\frac{d\pi^e}{dt} \qquad -(9)$$

From (5) $$p = \frac{1}{j}\frac{d\pi^e}{dt} + \pi^e \qquad \text{— (10)}$$

$$\frac{d^2\pi^e}{dt} = j\beta k m - j\beta k\left[\frac{1}{j}\frac{d\pi^e}{dt} + \pi^e\right] - j(1-g)\frac{d\pi^e}{dt}$$

$$\frac{d^2\pi^e}{dt^2} = j\beta k m - \beta k\frac{d\pi^e}{dt} - j\beta k\pi^e - j(1-g)\frac{d\pi^e}{dt}$$

$$\frac{d^2\pi^e}{dt} + \underbrace{\left[\beta k + j(1-g)\right]}_{a_1}\frac{d\pi^e}{dt} + \underbrace{j\beta k}_{a_2}\,\pi^e = \underbrace{j\beta k m}_{b} \qquad \text{— (11)}$$

$$y''(t) + a_1 y'(t) + a_2 y(t) = b$$

The Particular Integral

$$\left.\begin{aligned} \pi^e_p &= k_0 \\ \frac{d\pi^e}{dt} &= 0 \\ \frac{d^2\pi^e}{dt^2} &= 0 \end{aligned}\right\} \rightarrow (11)$$

$$0 + 0 + j\beta k(k_0) = j\beta k m$$

$$k_0 = m$$

$$\boxed{\pi^e_p = k_o = m}$$

Ex

$$p = \frac{1}{6} - 3U + \pi^e \qquad \text{— (i)} \quad ✓✓✓$$

$$\frac{d\pi^e}{dt} = \frac{3}{4}(p - \pi^e) \qquad \text{— (ii)} \quad ✓✓✓$$

$$\frac{dU}{dt} = -\frac{1}{2}(m - p) \qquad \text{— (iii)}$$

where, $\beta = 3$, $g = 1$, $j = \frac{3}{4}$, $k = \frac{1}{2}$

$$a_1 = \beta h + j(1-g) = \frac{3}{2}$$

$$a_2 = j\beta h = \frac{9}{8}$$

$$b = j\beta k m = \frac{9}{8}m$$

Note $\pi^e_p = \frac{b}{a_2} = m$

$$\frac{d^2\pi^e}{dt^2} + \frac{3}{2}\frac{d\pi^e}{dt} + \frac{9}{8}\pi^e = \frac{9}{8}m$$

Find y_c

solve the characteristic equations

$$r^2 + \frac{3}{2}r + \frac{9}{8} = 0$$

The characteristic roots are ....

$$r_1, r_2 = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2}$$

$$= -\frac{3}{4} \pm \frac{3}{4} i$$

$$h = \frac{-3}{4} \quad \text{and} \quad v = \frac{3}{4}$$

$$\pi_c^e = e^{ht}\left[A_5 \cos vt + A_6 \sin vt\right]$$

$$= e^{-\frac{3}{4}t}\left[A_5 \cos \frac{3}{4}t + A_6 \sin \frac{3}{4}t\right]$$

The general $sol^{\underline{n}}$ is

$$\pi^e(t) = \pi_c^e + \pi_p^e$$

$$= e^{-\frac{3}{4}t}\left[A_5 \cos \frac{3}{4}t + A_6 \sin \frac{3}{4}t\right] + m \quad \text{— (iv)}$$

The time path is damped fluctuation around the equilibrium value m. [b/c $h<0$]

Find $p(t)$ and $U(t)$

From (ii) $\qquad p = \frac{4}{3}\frac{d\pi^e}{dt} + \pi^e \quad$ — (v)