Solution: Quiz 3

1. Use mathematical induction proof to show that

$$2^n < n!$$

for all positive integer $n \geq 4$.

Answer:
Proof by mathematical induction:
Let $P(n)$ be the statement $2^n < n!$. We want to prove that $P(n)$ is true for any integer $n \geq 4$.

(I) **Basis step:** Show that $P(4)$ is true.
$P(4)$: $2^4 < 4!$.
Since $2^4 = 16$ and $4! = 1 \cdot 2 \cdot 3 \cdot 4 = 24$. Hence $2^4 < 4!$ and $P(4)$ is true.

(II) **Inductive step:** Show that if $P(k)$ is true, then $P(k+1)$ is also true, for any integer $k \geq 4$.

Assume that $P(k) : 2^k < k!$ is true. ——————($\bigstar$) "inductive hypothesis"
We want to show that $P(k+1) : 2^{k+1} < (k+1)!$ is true. Consider

$$\begin{aligned} 2^{k+1} &= 2^k \cdot 2 \\ &< k! \cdot 2 && \text{by } (\bigstar) \text{ "inductive hypothesis"} \\ &< k! \cdot (k+1) && \text{since } 2 < k+1 \text{ for } k \geq 4 \\ &= (k+1)! \end{aligned}$$

and therefore $P(k+1)$ is true. Note that we have used $k!(k+1) = \underbrace{1 \cdot 2 \cdot 3 \cdots k}_{k!} \cdot (k+1) = (k+1)!$.

From (I) basis step and (II) inductive step, $P(n)$ is true for all $n \geq 4$ by mathematical induction proof. ∎