







### Chapter 9

#### Optimization with Equality Constraint with many choice variables

-  Lagrange multiplier
-  Conditions for optimization
-  Maximize output level subject to cost constraint
-  Minimize cost subject to output constraint
-  ~~Minimize~~ Maximize utility subject to budget constraint
-  Minimize total expenditure under the level of utility



### Transition of idea

<p>Ch. 8</p> <p>Unconstrained optimization</p> <p>max/min <math>z = f(x_1, x_2)</math></p> <p><math>x_1, x_2</math></p>		<p>Ch. 9</p> <p>Constrained optimization</p> <p>max/min <math>z = f(x_1, x_2)</math></p> <p><math>x_1, x_2</math></p> <p>st. <math>h(x_1, x_2) = C \Rightarrow</math> constraint</p> <p>Lagrangian function</p> <p>max/min <math>L = f(x_1, x_2) + \lambda [C - h(x_1, x_2)]</math></p> <p><math>x_1, x_2, \lambda</math></p> <p>as if <math>\lambda</math> becomes another choice variable</p>	
<p><u>FOC</u></p> <p>partial derivative</p>	<p>w.r.t. every choice variables</p>	<p>matrix that collects every possible 2<sup>nd</sup> order partial derivative of obj <math>f_2</math> / Lagrangian <math>f_2</math></p> <p>Hessian Matrix</p>	<p>Bordereed Hessian matrix</p>

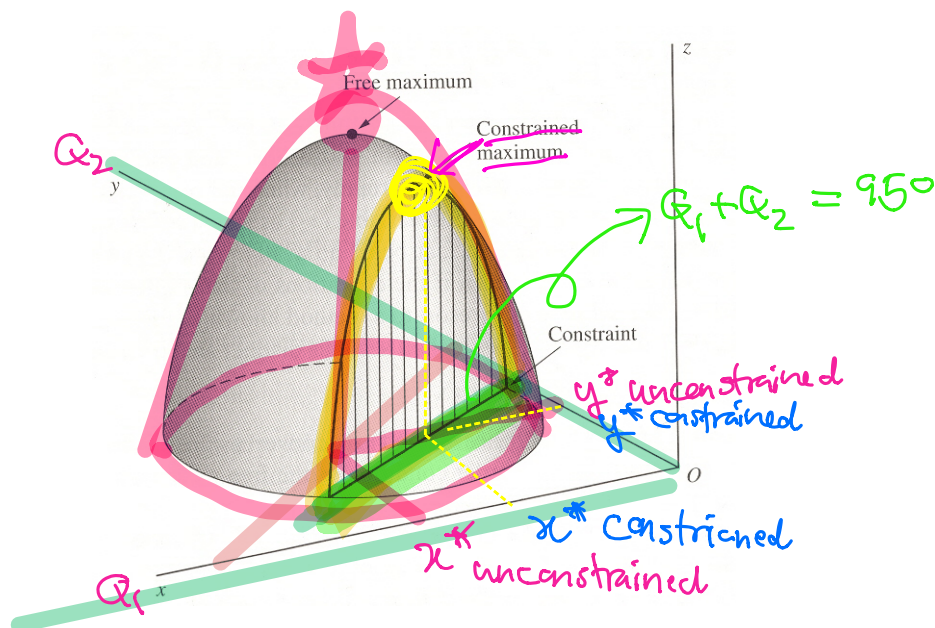
What if choice variables cannot be chosen freely? For example, what if a firm cannot freely choose quantities because of production quota? Firm will need to find constrained optimum value.

Suppose that a firm has two factories and is subjected to production quota of 950. The profit maximization problem becomes:

$$\begin{aligned} & \max_{Q_1, Q_2} \pi(Q_1, Q_2) \\ & \text{Subject to } Q_1 + Q_2 = 950 \end{aligned}$$

,where  $Q_1, Q_2$  are quantity produced at factory 1 and 2, respectively.

Free Extremum vs. Constrained Extremum



Economics is about finding the optimal allocation of scarce resources. Hence, the main problems in economics can be studied as constrained optimization problem. The prototype of such problem is:

maximize  $f(x_1, x_2, \dots, x_n)$   
 where  $(x_1, x_2, \dots, x_n)$  must satisfy

$g_1(x_1, x_2, \dots, x_n) \leq b_1$   
 $g_2(x_1, x_2, \dots, x_n) \leq b_2$   
 .....  
 $g_n(x_1, x_2, \dots, x_n) \leq b_n$

$h_1(x_1, x_2, \dots, x_n) = c_1$   
 $h_2(x_1, x_2, \dots, x_n) = c_2$   
 .....  
 $h_n(x_1, x_2, \dots, x_n) = c_n$

} inequality constraint

} equality constraint

what will be focusing on?



### Optimization with Equality Constraints : Two variables and One Equality Constraint

Let Mr. Musk have utility function  $U = x_1x_2 + 2x_1$ . If  $P_1 = 4$  and  $P_2 = 2$

Baht per piece, how many piece of good 1 and good 2 will Mr. Musk choose to consume to maximize his utility?

Suppose that Mr. Musk has no budget constraint

Max  $U = x_1x_2 + 2x_1$   
 $x_1, x_2$

FOC:  $U_{x_1} = x_2 + 2 = 0 \Rightarrow x_2^* = -2$   
 $U_{x_2} = x_1 = 0 \Rightarrow x_1^* = 0$

Sec:  $H = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$|H_1| = |0| = 0 \neq 0$ ,  $|H_2| = 0$  }  $H$  is indefinite

$|H_2| = -1 \neq 0$  }  $x_1^*, x_2^*$  don't give  $U_{max}$

$U_{x_1} = MU_{x_1} = x_2 + 2 \Rightarrow \frac{\partial MU_{x_1}}{\partial x_1} = 0$   
 $U_{x_2} = MU_{x_2} = x_1 \Rightarrow \frac{\partial MU_{x_2}}{\partial x_2} = 0$

} Law of diminishing MU doesn't apply here.



Lagrangian function for utility maximization problem when Mr. Musk has 60 baht is:

$$\max_{x_1, x_2, \mu} L = x_1 x_2 + 2x_1 + \mu [60 - 4x_1 - 2x_2]$$

these are there because of we consider info. from the constraint.

FONC:

$$L_\mu = \frac{\partial L}{\partial \mu} = 60 - 4x_1 - 2x_2 = 0 \Rightarrow 4x_1 + 2x_2 = 60 \quad (1)$$

$$L_1 = \frac{\partial L}{\partial x_1} = x_2 + 2 - 4\mu = 0 \Rightarrow x_2 + 2 = 4\mu \quad (2)$$

$$L_2 = \frac{\partial L}{\partial x_2} = x_1 - 2\mu = 0 \Rightarrow x_1 = 2\mu \quad (3)$$

$$(2) / (3) \quad \frac{x_2 + 2}{x_1} = \frac{4\mu}{2\mu} = 2 \Rightarrow x_1 = \frac{x_2 + 2}{2} \quad (4)$$

$$(4) \text{ in } (1) \quad 4 \left( \frac{x_2 + 2}{2} \right) + 2x_2 = 60$$

$$x_2^* = 14$$

$$x_1^* = 8$$

$$\mu^* = \frac{x_1^*}{2} = 4$$

Homework: Write Lagrangian function, first order conditions, and critical values of the following problems:

(a.)

$$\text{Max } U(x_1, x_2) = x_1 x_2$$

$$\text{St. } h(x_1, x_2) = x_1 + 4x_2 = 16$$

(b.)

$$\text{Max } z = xy$$

$$\text{St. } x + y = 6$$



### Optimization with Equality Constraints : n variables and One Equality Constraint

$$\begin{aligned} \max_{x_1, \dots, x_n} \quad & f(x_1, x_2, \dots, x_n) \\ \text{st.} \quad & h(x_1, x_2, \dots, x_n) = c \end{aligned}$$

Lagrangian function is

$$\max_{x_1, \dots, x_n, \mu} L = f(x_1, \dots, x_n) + \mu [c - h(x_1, \dots, x_n)]$$

= 0 if the const. is satisfied.

$$\begin{aligned} \text{FOC : } L_\mu &= \frac{\partial L}{\partial \mu} = c - h(x_1, \dots, x_n) = 0 \\ L_1 &= \frac{\partial L}{\partial x_1} = f_1 - \mu h_1 = 0 \\ L_2 &= \frac{\partial L}{\partial x_2} = f_2 - \mu h_2 = 0 \\ &\vdots \\ L_n &= \frac{\partial L}{\partial x_n} \end{aligned}$$

Use n+1 FOCs to solve for critical values of  $x_1^*, x_2^*, \dots, x_n^*, \mu^*$

Note 1 : At critical values, the constraint is satisfied,

$$L^* = f(x_1^*, x_2^*, \dots, x_n^*) + \mu^* [c - h(x_1^*, x_2^*, \dots, x_n^*)]$$

$$L^* = f(x_1^*, x_2^*, \dots, x_n^*) \quad \mu^* = 0$$

The value of Lagrangian  $L^*$  is equal to the value of obj  $f^*$  at the critical point.

$$\text{Note 2 : } \frac{\partial L^*}{\partial c} = \mu^* = \frac{\partial f(x_1^*, \dots, x_n^*)}{\partial c}$$

shows how optimized value of objective  $f^*$  changes when constraint  $c$  changes 1 unit

ex. when budget increases by 1 bcht, how much will the maximized utility change?  $\Rightarrow$  it would change by  $\mu^*$  utils.

### ∞ Second-Order Sufficient Condition ∞

Consider objective function:

$$\begin{aligned} & \text{Max}_{x_1, \dots, x_n} f(x_1, x_2, \dots, x_n) \\ \text{St. } & h^1(x_1, x_2, \dots, x_n) = c_1 \\ & h^2(x_1, x_2, \dots, x_n) = c_2 \\ & \dots \\ & h^k(x_1, x_2, \dots, x_n) = c_k \end{aligned}$$

} k equality const.  
} k lagrange multipliers.

The Lagrangian

$$\text{max}_{x_1, \dots, x_n, \mu_1, \dots, \mu_k} L(x_1, x_2, \dots, x_n) = f(x_1, x_2, \dots, x_n) + \mu_1 [c_1 - h^1(x_1, x_2, \dots, x_n)] + \mu_2 [c_2 - h^2(x_1, x_2, \dots, x_n)] + \dots + \mu_k [c_k - h^k(x_1, x_2, \dots, x_n)]$$

} k terms

} n+k  
} n choice vars } k lagrange multipliers

First Order Necessary Condition: FONC

$$\left. \begin{aligned} L_{x_1} = \frac{\partial L}{\partial x_1} = 0 & \dots f_1 - \mu_1 h_1^1 - \mu_2 h_1^2 - \dots - \mu_k h_1^k = 0 \\ \dots \\ L_{x_n} = \frac{\partial L}{\partial x_n} = 0 & \dots f_n - \mu_1 h_n^1 - \mu_2 h_n^2 - \dots - \mu_k h_n^k = 0 \\ \dots \\ L_{\mu_1} = \frac{\partial L}{\partial \mu_1} = 0 & \dots c_1 - h^1(x_1, \dots, x_n) = 0 \\ \dots \\ L_{\mu_k} = \frac{\partial L}{\partial \mu_k} = 0 & \dots c_k - h^k(x_1, \dots, x_n) = 0 \end{aligned} \right\} \begin{array}{l} n \text{ eqs.} \\ k \text{ eqs.} \end{array}$$

} n+k eqs.

Second Order Sufficient Condition: SOSC

“Bordered Hessian Matrix”.....  $\bar{H}$  : a matrix collecting every possible 2<sup>nd</sup> order partial derivatives of the lagrangian f<sup>n</sup>

1<sup>st</sup> order partial derivatives :  $L_{\mu_1}, L_{\mu_2}, \dots, L_{\mu_k}, L_{x_1}, L_{x_2}, \dots, L_{x_n}$  : n+k terms.  
Take partial diff of 1<sup>st</sup> order w.r.t. :  $\mu_1, \dots, \mu_k, x_1, \dots, x_n$   
We can divide Bordered Hessian Matrix into 4 parts:

Note: We evaluate Bordered Hessian at the critical values.

	Col 1	Col 2	...	Col k	...	Col k+1	Col k+2	...	Col k+n	
row 1	$L_{\mu_1 \mu_1}$	$L_{\mu_1 \mu_2}$	...	$L_{\mu_1 \mu_k}$	...	$L_{\mu_1 x_1}$	$L_{\mu_1 x_2}$	...	$L_{\mu_1 x_n}$	partial diff w.r.t. $\mu_1$
row 2	$L_{\mu_2 \mu_1}$	$L_{\mu_2 \mu_2}$	...	$L_{\mu_2 \mu_k}$	...	$L_{\mu_2 x_1}$	$L_{\mu_2 x_2}$	...	$L_{\mu_2 x_n}$	$\mu_2$
...	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
row k	$L_{\mu_k \mu_1}$	$L_{\mu_k \mu_2}$	...	$L_{\mu_k \mu_k}$	...	$L_{\mu_k x_1}$	$L_{\mu_k x_2}$	...	$L_{\mu_k x_n}$	$\mu_k$
row k+1	$L_{x_1 \mu_1}$	$L_{x_1 \mu_2}$	...	$L_{x_1 \mu_k}$	...	$L_{11}$	$L_{12}$	...	$L_{1n}$	$x_1$
...	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$x_2$
row k+n	$L_{x_n \mu_1}$	$L_{x_n \mu_2}$	...	$L_{x_n \mu_k}$	...	$L_{n1}$	$L_{n2}$	...	$L_{nn}$	$x_n$

**Hessian**

(n+k) × (n+k)

$L_{\mu_i \mu_j} = 0 \quad \begin{matrix} i=1, \dots, k \\ j=1, \dots, k \end{matrix}$

~ taking partial diff of the first const w.r.t.  $x_1, \dots, x_n$  and with negative sign

$$\bar{H} = \begin{bmatrix} 0 & 0 & \dots & 0 & -h_1^1 & -h_2^1 & \dots & -h_n^1 \\ 0 & 0 & \dots & 0 & -h_1^2 & -h_2^2 & \dots & -h_n^2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & -h_1^k & -h_2^k & \dots & -h_n^k \\ -h_1^1 & -h_1^2 & \dots & -h_1^k & L_{11} & L_{12} & \dots & L_{1n} \\ -h_2^1 & -h_2^2 & \dots & -h_2^k & L_{21} & L_{22} & \dots & L_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -h_n^1 & -h_n^2 & \dots & -h_n^k & L_{n1} & L_{n2} & \dots & L_{nn} \end{bmatrix}_{(n+k) \times (n+k)}$$

We can have submatrix  $H_1$  with border,  $\bar{H}_1$ , and so on. Try to get the submatrix from  $\bar{H}$ .

$\bar{H}_1$  is  $H_1$  including all rows above & all column to the left

$$\bar{H}_1 = \begin{bmatrix} 0 & 0 & \dots & 0 & -h_1^1 & -h_2^1 & \dots & -h_n^1 \\ 0 & 0 & \dots & 0 & -h_1^2 & -h_2^2 & \dots & -h_n^2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & -h_1^k & -h_2^k & \dots & -h_n^k \\ -h_1^1 & -h_1^2 & \dots & -h_1^k & L_{11} & L_{12} & \dots & L_{1n} \\ -h_2^1 & -h_2^2 & \dots & -h_2^k & L_{21} & L_{22} & \dots & L_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -h_n^1 & -h_n^2 & \dots & -h_n^k & L_{n1} & L_{n2} & \dots & L_{nn} \end{bmatrix}_{(n+k) \times (n+k)}$$

$\bar{H}_2$

$$\bar{H}_2 = \begin{bmatrix} 0 & 0 & \dots & 0 & -h_1^1 & -h_2^1 & \dots & -h_n^1 \\ 0 & 0 & \dots & 0 & -h_1^2 & -h_2^2 & \dots & -h_n^2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & -h_1^k & -h_2^k & \dots & -h_n^k \\ -h_1^1 & -h_1^2 & \dots & -h_1^k & L_{11} & L_{12} & \dots & L_{1n} \\ -h_2^1 & -h_2^2 & \dots & -h_2^k & L_{21} & L_{22} & \dots & L_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -h_n^1 & -h_n^2 & \dots & -h_n^k & L_{n1} & L_{n2} & \dots & L_{nn} \end{bmatrix}_{(n+k) \times (n+k)}$$

**Question:**  
 How many submatrices do we need to consider?

**Answer:**  
 $n - k$  matrices, i.e. the last  $n-k$  leading principle minors of matrix  $\bar{H}$ .

$n$  is the number of choice variables and  $k$  is the number of constraints.

We need to focus since  $\bar{H}_{k+1}$ .

We need to check:

$$\bar{H}_{k+1}, \bar{H}_{k+2}, \dots, \bar{H}_n = \bar{H}$$

$\bar{H}_1, \bar{H}_2, \bar{H}_3, \dots, \bar{H}_k$

$\bar{H}_{k+1}, \dots, \bar{H}_n$

drop off from our consideration      consider the last  $n-k$  submatrices

Example 1: the case of one constraint with two choice variables:

$$\bar{H} = \begin{bmatrix} 0 & -h_{x_1} & -h_{x_2} \\ -h_{x_1} & L_{x_1 x_1} & L_{x_1 x_2} \\ -h_{x_2} & L_{x_2 x_1} & L_{x_2 x_2} \end{bmatrix}$$

$3 \times 3$   
 $(2+1) \times (2+1)$

Number of choice variables (n) ..... 2

Number of constraint (k) ..... 1

So we need to check the last n-k = ..... 2-1 = 1 ..... Matrix, which is:

$$\bar{H}_{k+1} = \bar{H}_{1+1} = \bar{H}_2 = \begin{bmatrix} 0 & -h_{x_1} & -h_{x_2} \\ -h_{x_1} & L_{x_1x_1} & L_{x_1x_2} \\ -h_{x_2} & L_{x_2x_1} & L_{x_2x_2} \end{bmatrix} = \bar{H}$$

Example 2: 3 choice variables and 1 constraint

$$\bar{H} = \begin{bmatrix} 0 & -h_{x_1} & -h_{x_2} & -h_{x_3} \\ -h_{x_1} & L_{x_1x_1} & L_{x_1x_2} & L_{x_1x_3} \\ -h_{x_2} & L_{x_2x_1} & L_{x_2x_2} & L_{x_2x_3} \\ -h_{x_3} & L_{x_3x_1} & L_{x_3x_2} & L_{x_3x_3} \end{bmatrix} \quad (3+1) \times (3+1)$$

Annotations:  $\bar{H}_1$  is circled in pink and labeled "drop".  $\bar{H}_2$  and  $\bar{H}_3$  are circled in cyan and labeled "keep".

Number of choice variables (n) ..... 3

Number of constraint (k) ..... 1

keep the last n-k, drop the first k

So we need to check the last n-k = ..... 3-1 = 2 ..... Matrix, beginning at  $\bar{H}_{k+1} = \bar{H}_{1+1} = \bar{H}_2$

$$\bar{H}_2 = \begin{bmatrix} 0 & -h_{x_1} & -h_{x_2} \\ -h_{x_1} & L_{x_1x_1} & L_{x_1x_2} \\ -h_{x_2} & L_{x_2x_1} & L_{x_2x_2} \end{bmatrix} \quad \bar{H}_3 = \bar{H} = \begin{bmatrix} 0 & -h_{x_1} & -h_{x_2} & -h_{x_3} \\ -h_{x_1} & L_{x_1x_1} & L_{x_1x_2} & L_{x_1x_3} \\ -h_{x_2} & L_{x_2x_1} & L_{x_2x_2} & L_{x_2x_3} \\ -h_{x_3} & L_{x_3x_1} & L_{x_3x_2} & L_{x_3x_3} \end{bmatrix}$$

We need to observe pattern of the determinants of leading principal submatrices as the following.

The objective function will be at the local max if the last  $(n-k)$  leading principal minors have alternate signs with  $|\bar{H}_{k+1}|$  has the same sign as  $(-1)^{k+1}$ .

The objective function will be at the local min if the last  $(n-k)$  leading principal minors have the same sign as  $(-1)^k$  for every bordered Hessian that we need to consider.

∞ SOC of 2 choice variables + 1 equality constraint ∞

$$|\bar{H}_2| = \begin{vmatrix} 0 & -h_{x_1} & -h_{x_2} \\ -h_{x_1} & L_{x_1x_1} & L_{x_1x_2} \\ -h_{x_2} & L_{x_2x_1} & L_{x_2x_2} \end{vmatrix} \quad 3 \times 3$$

If  $|\bar{H}_2| > 0 \quad \rightarrow$

If  $|\bar{H}_2| < 0 \quad \rightarrow$

$n = 2, k = 1$  (consider  $x_1, x_2, \mu$  when taking partial diff of Lagrangian  $f^2$ )

look at the last  $n-k = 2-1 = 1$  matrix

which is  $\bar{H}_{k+1} = \bar{H}_{1+1} = \bar{H}_2$

Obj.  $f^2$  is maximized when  $|\bar{H}_{k+1}|$  has the same sign as  $(-1)^{k+1} = (-1)^{1+1} = (-1)^2 = 1 > 0$

$\therefore |\bar{H}_{k+1}| = |\bar{H}_2| > 0$

minimized

has the same sign as

$(-1)^k = (-1)^1 = -1 < 0$

$|\bar{H}_{k+1}| < 0$

"  $|\bar{H}_2|$

∞ FOC, SOC ∞ n choice variables + 1 equality constraint ∞

$$\begin{aligned} & \max / \min f(x_1, x_2, \dots, x_n) \\ & \text{St. } h(x_1, \dots, x_n) = c \\ & \quad g(x_1, x_2, \dots, x_n) = c \end{aligned}$$

FOC:  $L = f(x_1, x_2, \dots, x_n) + \lambda [c - g(x_1, x_2, \dots, x_n)]$

$L_1 = L_2 = L_3 = \dots = L_n = L_\lambda = 0$

SOC:

★ MAX

$|\bar{H}_2| > 0; |\bar{H}_3| < 0; |\bar{H}_4| > 0; \dots$

look at the last  $n-k = n-1$  leading submatrices, starting to look at  $\bar{H}_{k+1} = \bar{H}_2$

$|\bar{H}_2|$

has the same sign as  $(-1)^{k+1} = (-1)^{1+1} > 0$  and alternate

★ MIN

$|\bar{H}_2|; |\bar{H}_3|; |\bar{H}_4|; \dots; |\bar{H}_n| < 0$

$|\bar{H}_2|$

has the same sign as  $(-1)^k = (-1)^1 < 0$

$\bar{H}_2$     $\bar{H}_3$     $\bar{H}_n = \bar{H}$

$$\bar{H} = \begin{bmatrix} 0 & -h_{x_1} & -h_{x_2} & -h_{x_3} & \dots & -h_{x_n} \\ -h_{x_1} & L_{11} & L_{12} & L_{13} & \dots & L_{1n} \\ -h_{x_2} & L_{21} & L_{22} & L_{23} & \dots & L_{2n} \\ -h_{x_3} & L_{31} & L_{32} & L_{33} & \dots & L_{3n} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -h_{x_n} & L_{n1} & L_{n2} & L_{n3} & \dots & L_{nn} \end{bmatrix}$$

$(n+1) \times (n+1)$

Max Q  
K, L  
st. C

Min TC  
K, L  
st. Q<sub>0</sub>



Max U  
x, y  
st. B

Min Exp  
x, y  
st. U<sub>0</sub>

Output maximization problem

Cost minimization problem (CMP)

Utility Maximization Problem (UMP)

Expenditure minimization Problem (EMP)

Maximizing output level subject to cost constraint  
Let the production function be  $Q = q(L, K)$

The total cost is:

$$TC = wL + rK = C$$

Question: How will firm choose  $L$  and  $K$  to maximize output level, given that the total cost must be  $C$  baht.

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**Step 1: Lagrangian Function**

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**Step 2: First-Order Condition (FONC):**

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**Minimizing cost given a level of output:**

**Least Cost Combination of Inputs and Conditional Input Demand**

Assume that a firm needs to produce output equal to  $Q_0$  units from production function  $Q = F(L, K)$ , and the firm would like to choose the level of capital and labor that minimize the cost.

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**Step 1: Lagrangian Function**

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**Step 2: First-Order Condition (FONC):**

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**3.2 Check the last n-k leading principle Minor**

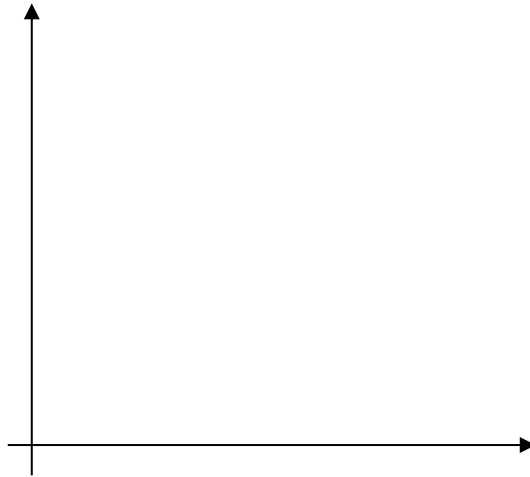
n = \_\_\_\_\_ k = \_\_\_\_\_

We need to check the last \_\_\_\_\_ leading principle Minor นั่นคือ

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**The Expansion Path**

Expansion Path is the least cost combination between L\* and K\* given many levels of given outputs.



If Isoquants are strictly convex, expansion path can be found from the first order conditions.

If a firm has Cobb-Douglas production function  $Q = AL^\alpha K^\beta$ ,

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In this case, the optimal ratio of capital to labor is constant and is equal to \_\_\_\_\_







**Utility maximization subject to budget constraint**

$$U = U(x, y) \quad (U_x, U_y > 0)$$

$$\text{St. } xP_x + yP_y = B$$

**Step 1: Lagrangian Function**

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**Step 2: First-Order Condition (FONC):**

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**Economic Interpretation**

**Interpretation 1 : Marginal utility per one baht paid**

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**Interpretation 2: Indifference Curve**



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**Meaning of Lagrange multiplier**

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**Step 3 Second-Order Sufficient Condition**

**3.1 Construct Bordered Hessian**

$$\bar{H} = \begin{bmatrix} L_{\lambda\lambda} = & L_{\lambda x} = & L_{\lambda y} = \\ L_{x\lambda} = & L_{xx} = \frac{\partial^2 U}{\partial x^2} = & L_{xy} = \frac{\partial^2 U}{\partial x \partial y} = \\ L_{y\lambda} = & L_{yx} = \frac{\partial^2 U}{\partial y \partial x} = & L_{yy} = \frac{\partial^2 U}{\partial y^2} = \end{bmatrix}$$

$$\bar{H} = \begin{bmatrix} \\ \\ \\ \end{bmatrix}$$

**3.2 Check the last n-k leading principle Minor**

n = \_\_\_\_\_ k = \_\_\_\_\_

Check the last \_\_\_\_\_ leading principle Minor :

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**Minimize Total Expenditure Under the Level of Utility**

Suppose now that a consumer would like to have utility  $U_0$ . Her utility is  $U = U(x, y)$ .

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**Step 1 Lagrangian Function**

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### Duality: Utility Maximization Problem vs. Expenditure Minimization Problem

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### Other optimization problems:

#### Multiproduct firm with technological constraint

Quantities of two products are  $x$  &  $y$ .

How many of  $x$  &  $y$  should the firm choose to produce to minimize the cost of production subjecting to technological constraint?

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### Multiplicant firm who knows exact total market demand

Let factory 1 produce  $Q_1$  with cost  $C_1 = C_1(Q_1)$ ; factory 2 produce  $Q_2$  with cost  $C_2 = C_2(Q_2)$ ; and let market demand be  $Q_0$ .

To minimize cost, how many should the firm produce at each factory given that it wants to meet the market demand?

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