

EE320 (1/2017)

INTRODUCTORY MATHEMATICAL ECONOMICS

OPTIMIZATION UNDER EQUALITY CONSTRAINTS

Topics

- Effects of a constraint
- Finding the stationary (optimal) values
 - Elimination method
 - Lagrange multiplier
 - n -variable and multi-constraint cases
- Second-order conditions for constrained optimization
- Economic applications
- Extensions: n -variable and multiconstraint cases

Effects of a Constraint

- In the previous topic, we studied optimization problems, where the choice variables can be chosen freely.
- Example: A profit-maximizing firm's objective is:

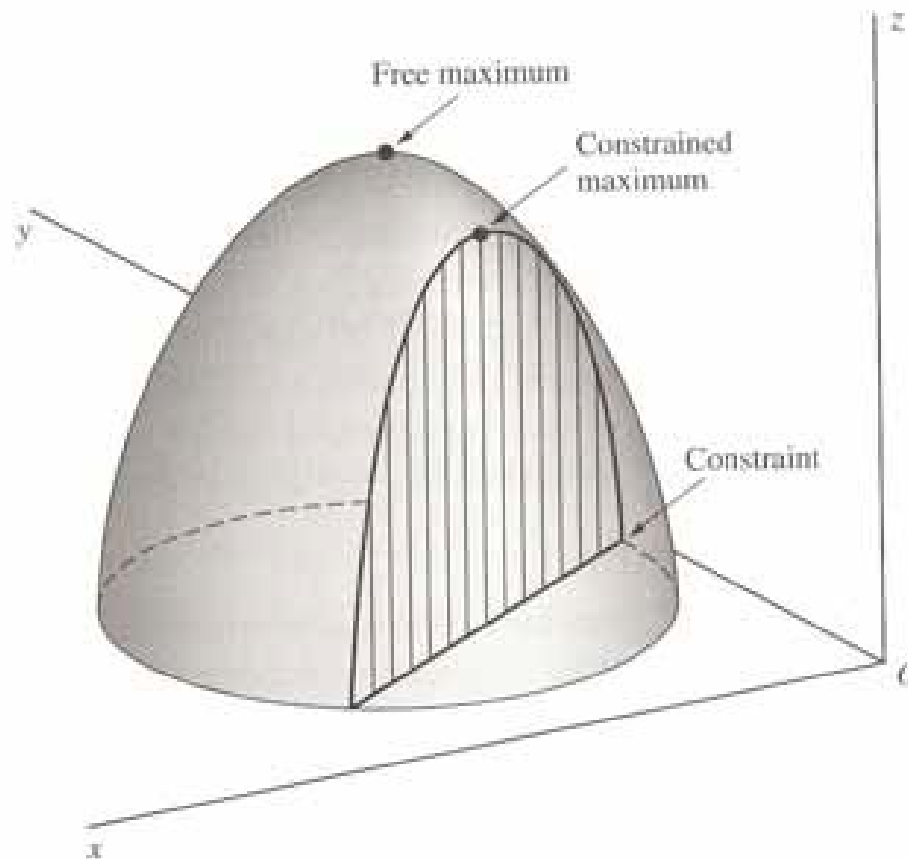
$$\pi = PQ - [C(Q_1) + C(Q_2)]$$

where $Q = Q_1 + Q_2$.

- Q_1 and Q_2 can be chosen freely in order to maximize profit.
→ Here, Q_1^* and Q_2^* are *free optimal values*.
- Question: If there is a *production constraint*, e.g. $Q_1 + Q_2 = 900$, then how would the firm change its behavior?
- → Q_1^{**} and Q_2^{**} will now become the *constrained optimal values*.

Effects of a Constraint

- **Figure:** “Free extremum” vs. “Constrained extremum”



FINDING THE STATIONARY VALUES

Two Choice Variables with One Equality Constraint

➤ Use *elimination* (or *substitution*) method

Example: $\max_{x_1, x_2} U = x_1 x_2 + 2x_1$
subject to $4x_1 + 2x_2 = 60$

Two Choice Variables with One Constraint

➤ Use *Lagrange-multiplier* method

Consider the following maximization problem.

$$\begin{array}{ll} \max_{x,y} & f(x,y) \\ & \text{subject to } g(x,y) = c \end{array} .$$

Define the *Lagrangian function* by

F.O.N.C.

Example 1: 2 choice variables & 1 constraint

- Find the optimal values for the following function.

$$U = x_1x_2 + 2x_1$$

subject to $4x_1 + 2x_2 = 60$

Step 1- Set up the Lagrangian function:

Step 2 - Find FONC:

Example 2: 2 choice variables & 1 constraint

- Find the optimal values for the following function:

$$f(x_1, x_2) = x_1^2 + x_2^2$$

$$\text{subject to } g(x_1, x_2) = x_1 + 4x_2 = 2$$

Step 1 - Set up the Lagrangian function:

Step 2 - Find FONC:

Interpretation of Lagrange Multiplier

- Consider the problem

$$\max_{x,y} (\min) f(x, y) \text{ subject to } g(x, y) = c$$

- Solutions to this problem: $x^*(c)$, $y^*(c)$, and $f^*(c)$.
- $f^*(c)$ is called **optimal value** function.
- We can show that $\frac{df^*(c)}{dc} = \lambda(c)$
 - The **Lagrange multiplier** λ is the rate at which the optimal value of the objective function changes with respect to changes in the constraint constant, c .

Interpretation of Lagrange Multiplier

- In economic applications, c denotes the available stock of some resources, $f(x, y)$ denotes utility or profit.
- $\lambda(c)dc$ is the change in utility or profit that can be obtained from dc units more of the resource.
- Thus, λ is called a “shadow price” of the resource.
- Examples:
 - For cost minimization problem, $\lambda = \frac{dC}{dQ_0} = MC$
 - For utility maximization problem, $\lambda = \frac{dU}{dY_0} = \text{MU of income}$
 - For profit maximization problem, $\lambda = \frac{d\pi}{dQ_0}$
 - For output maximization problem, $\lambda = \frac{dQ}{dC_0}$

SECOND-ORDER CONDITIONS FOR CONSTRAINED OPTIMIZATION

Second-Order Total Differential

- Recall the FONC for $L(x, y, \lambda) = f(x, y) + \lambda[c - g(x, y)]$

$$L_x(x, y) = f_x(x, y) - \lambda g_x(x, y) = 0$$

$$L_y(x, y) = f_y(x, y) - \lambda g_y(x, y) = 0$$

- By partially differentiating the FONC, the second derivatives are:

$$L_{xx}$$

$$L_{xy}$$

$$L_{yy}$$

- Thus, d^2z can be rewritten as:

$$d^2z =$$

Second-Order Conditions

- The **second-order sufficient conditions** are:
 - For *maximum* of z : d^2z is *negative definite*, subject to $dg = 0$
 - For *minimum* of z : d^2z is *positive definite*, subject to $dg = 0$
 where $dg = g_x dx + g_y dy = 0$.
- Define the determinant of the *Bordered Hessian matrix* as:

where $g_x = \partial g / \partial x$; $g_y = \partial g / \partial y$;

and L_{xx} , L_{xy} , L_{yx} , L_{yy} are the second-order derivatives of $L(x, y, \lambda)$.

Second-Order Conditions

- The **sign definiteness of d^2z** can be determined from the following criterion:
 - **d^2z is negative definite** subject to $g_x dx + g_y dy = 0$ iff $|\bar{H}| > 0$.
 - **d^2z is positive definite** subject to $g_x dx + g_y dy = 0$ iff $|\bar{H}| < 0$.

- Summary:

For a stationary value of $L(x, y) = f(x, y) + \lambda[c - g(x, y)]$

The **second-order sufficient conditions** are:

For **maximum** of z , the ***bordered Hessian is positive*** ($|\bar{H}| > 0$);

For **minimum** of z , the ***bordered Hessian is negative*** ($|\bar{H}| < 0$).

Example

- Determine whether the extremum of the objective function

$$U = x_1x_2 + 2x_1 \quad \text{subject to} \quad 4x_1 + 2x_2 = 60$$

gives a minimum or a maximum.

Example

- Determine whether the extremum of the objective function

$$f(x_1, x_2) = x_1^2 + x_2^2 \quad \text{subject to} \quad g(x_1, x_2) = x_1 + 4x_2 = 2$$

gives a minimum or a maximum.

ECONOMIC APPLICATIONS OF CONSTRAINED OPTIMIZATION

Utility Maximization Problem

- Objective function: $\max_{Q_1, Q_2} U = U(Q_1, Q_2)$
- Subject to: $Y_0 = p_1 Q_1 + p_2 Q_2$
- Lagrangian function:

Utility Maximization Problem

- Second-order sufficient condition

Utility Maximization Problem

- **Example:** Suppose that $U = 50A + 200B - 0.5A^2 - 2.5B^2$.
Find A and B that maximize the utility given that $P_A = 10$, $P_B = 5$,
and $Y = 490$.

Expenditure Minimization

- Objective function: $\underset{Q_1, Q_2}{Min} E = p_1 Q_1 + p_2 Q_2$
- Subject to: $\bar{U} = U(Q_1, Q_2)$
- Lagrangian function:

Expenditure Minimization

- **Example:** Suppose that $U = 50A + 200B - 0.5A^2 - 2.5B^2$, and the consumer would like to have a utility fixed at U_0 . Find A and B that minimize the expenditure given that $P_A = 10$ and $P_B = 5$.

Profit Maximization Problem

- Objective function: $\max_{Q_1, Q_2} \pi = p_1 Q_1 + p_2 Q_2 - c(Q_1, Q_2)$
- Subject to: $Q_0 = Q_1 + Q_2$
- Lagrangian function:

Profit Maximization Problem

- **Example:** Suppose that $P_1 = 80$, $P_2 = 100$, $TC = 100 + 0.1Q_1^2 + 0.2Q_2^2$
And $Q_1 + Q_2 = 325$. Find Q_1 and Q_2 that maximize the profit.

Output Maximization Problem

- Objective function: $\max_{K,L} Q = Q(K,L)$
- Subject to: $C_0 = wL + rK$
- Lagrangian function:

Output Maximization Problem

- **Example:** Suppose $Q = KL$, $w = 6$, $r = 10$, $C_0 = 60$. Find K and L that maximizes output.

Cost Minimization

- Objective function: $\underset{K,L}{Min} C = wL + rK$
- Subject to: $Q_0 = f(K, L)$
- Lagrangian function:

Cost Minimization

- Graph: Expansion path

Cost Minimization

- **Example:** Suppose $Q = KL$. A firm will produce 15 units of the good. Find K and L that minimize total cost given that $w = 6$, $r = 10$.

EXTENSION: N-CHOICE VARIABLE AND MULTI-CONSTRAINT CASES

Optimization with Equality Constraints: *n*-Variable and One Equality Constraint

- The objective function

$$z = f(x_1, x_2, \dots, x_n)$$

Subject to the constraint $g(x_1, x_2, \dots, x_n) = c$

The Lagrangian function:

$$L(x_1, \dots, x_n, \lambda) = f(x_1, \dots, x_n) + \lambda[c - g(x_1, \dots, x_n)]$$

FONC:

$$L_1 = f_1(x_1, \dots, x_n) - \lambda g_1(x_1, \dots, x_n) = 0$$

$\vdots$

$$L_n = f_n(x_1, \dots, x_n) - \lambda g_n(x_1, \dots, x_n) = 0$$

$$L_\lambda(x_1, \dots, x_n) = c - g(x_1, \dots, x_n) = 0$$

Second-Order Conditions:

n -Variable and One Equality Constraint

- The objective function $z = f(x_1, x_2, \dots, x_n)$
 Subject to $g(x_1, x_2, \dots, x_n) = c$

➤ **Bordered Hessian:**

$$|\overline{H}| = \begin{vmatrix} 0 & g_1 & g_2 & \cdots & g_n \\ g_1 & L_{11} & L_{12} & \cdots & L_{1n} \\ g_2 & L_{21} & L_{22} & \cdots & L_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ g_n & L_{n1} & L_{n2} & \cdots & L_{nn} \end{vmatrix}$$

- **Bordered leading principal minors** can be defined as:

$$|\overline{H}_2| = \begin{vmatrix} 0 & g_1 & g_2 \\ g_1 & L_{11} & L_{12} \\ g_2 & L_{21} & L_{22} \end{vmatrix}; |\overline{H}_3| = \begin{vmatrix} 0 & g_1 & g_2 & g_3 \\ g_1 & L_{11} & L_{12} & L_{13} \\ g_2 & L_{21} & L_{22} & L_{23} \\ g_3 & L_{31} & L_{32} & L_{33} \end{vmatrix}; \dots; |\overline{H}_n| = |\overline{H}|$$

(Note: The subscript refers to the size of the bordered matrix.)

Second-Order Conditions: *n*-Variable and One Equality Constraint

- The conditions for positive and negative definiteness of d^2z are:

$$d^2z \text{ is } \begin{cases} \text{positive_definite} \\ \text{negative_definite} \end{cases} \text{ subject to } dg = 0 \text{ iff } \begin{cases} |\overline{H}_2|, |\overline{H}_3|, \dots, |\overline{H}_n| < 0 \\ |\overline{H}_2| > 0; |\overline{H}_3| < 0; |\overline{H}_4| > 0; \text{etc.} \end{cases}$$

Condition	Maximum	Minimum
First-order necessary condition	$L_\lambda = L_1 = L_2 = \dots = L_n = 0$	$L_\lambda = L_1 = L_2 = \dots = L_n = 0$
Second-order necessary condition*	$ \overline{H}_2 > 0; \overline{H}_3 < 0;$ $ \overline{H}_4 > 0; \dots; (-1)^n \overline{H}_n > 0$	$ \overline{H}_2 , \overline{H}_3 , \dots, \overline{H}_n < 0$

Cost Minimization

- Objective function: $\underset{K,L,T}{Min} C = wL + iK + rT$
- Subject to: $Q_0 = f(K, L, T)$
- Lagrangian function:

Optimization with Equality Constraints: n -Variable and Two Equality Constraints

- The objective function

$$z = f(x_1, x_2, \dots, x_n)$$

Subject to the constraints $g(x_1, x_2, \dots, x_n) = c$
 $h(x_1, x_2, \dots, x_n) = d$

➤ **The Lagrangian function:**

$$L = f(x_1, x_2, \dots, x_n) + \lambda_1 [c - g(x_1, x_2, \dots, x_n)] + \lambda_2 [d - h(x_1, x_2, \dots, x_n)]$$

➤ **FONC:**

Optimization with Equality Constraints:

n -Variable and k Equality Constraints

- The objective function

$$z = f(x_1, x_2, \dots, x_n)$$

Subject to the constraints

$$g_1(x_1, x_2, \dots, x_n) = c_1$$

$$g_2(x_1, x_2, \dots, x_n) = c_2$$

.....

$$g_k(x_1, x_2, \dots, x_n) = c_k$$

➤ The Lagrangian function:

$$L = f(x_1, x_2, \dots, x_n) + \lambda_1 [c_1 - g^1(x_1, x_2, \dots, x_n)] + \lambda_2 [c_2 - g^2(x_1, x_2, \dots, x_n)]$$

$$+ \dots + \lambda_k [c_k - g^k(x_1, x_2, \dots, x_n)]$$

Optimization with Equality Constraints:

n -Variable and k Equality Constraints

➤ **FONC:**

$$\frac{\partial L}{\partial x_1} = f_1(x_1, \dots, x_n) - \lambda_1 g_1^1(x_1, \dots, x_n) - \dots - \lambda_k g_1^k(x_1, \dots, x_n) = 0$$

$$\frac{\partial L}{\partial x_2} = f_2(x_1, \dots, x_n) - \lambda_1 g_2^1(x_1, \dots, x_n) - \dots - \lambda_k g_2^k(x_1, \dots, x_n) = 0$$

.....

$$\frac{\partial L}{\partial x_n} = f_n(x_1, \dots, x_n) - \lambda_1 g_n^1(x_1, \dots, x_n) - \dots - \lambda_k g_n^k(x_1, \dots, x_n) = 0$$

$$\frac{\partial L}{\partial \lambda_1} = c_1 - g^1(x_1, \dots, x_n) = 0$$

$$\frac{\partial L}{\partial \lambda_2} = c_2 - g^2(x_1, \dots, x_n) = 0$$

.....

$$\frac{\partial L}{\partial \lambda_k} = c_k - g^k(x_1, \dots, x_n) = 0$$

Second-Order Conditions:

n -Variable and k Equality Constraints

The Lagrangian condition is:

$$L = f(x_1, x_2, \dots, x_n) + \sum_{j=1}^k \lambda_j [c_j - g^j(x_1, x_2, \dots, x_n)]$$

➤ Bordered Hessian:

$$|\overline{H}| \equiv \begin{vmatrix} 0 & 0 & \cdots & 0 & g_1^1 & g_2^1 & \cdots & g_n^1 \\ 0 & 0 & \cdots & 0 & g_1^2 & g_2^2 & \cdots & g_n^2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & g_1^k & g_2^k & \cdots & g_n^k \\ \hline g_1^1 & g_1^2 & \cdots & g_1^k & L_{11} & L_{12} & \cdots & L_{1n} \\ g_2^1 & g_2^2 & \cdots & g_2^k & L_{21} & L_{22} & \cdots & L_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ g_n^1 & g_n^2 & \cdots & g_n^k & L_{n1} & L_{n2} & \cdots & L_{nn} \end{vmatrix}$$

Second-Order Conditions:

n -Variable and k Equality Constraints

- The second-order sufficient conditions are stated in terms of the signs of the following $(n-k)$ bordered leading principal minors:

$$|\overline{H}_{k+1}|, |\overline{H}_{k+2}|, \dots, |\overline{H}_n| \quad \text{where } |\overline{H}_n| = |\overline{H}|$$

- SOSC:

- For a *maximum of z* , the bordered leading principal minors *alternate in sign*, and the sign of $|\overline{H}_{k+1}| = (-1)^{k+1}$.
- For a *minimum of z* , all the bordered leading principal minors *take the same sign*, namely that of $(-1)^k$.

Example

- The objective function

$$z = f(x_1, x_2, x_3)$$

$$\text{Subject to } g(x_1, x_2, x_3) = c$$

$$h(x_1, x_2, x_3) = d$$

- The Lagrangian function:

$$L = f(x_1, x_2, x_3) + \lambda_1 [c - g(x_1, x_2, x_3)] + \lambda_2 [d - h(x_1, x_2, x_3)]$$

- FONC:

- SOSC: