Topic 4 : Capital Asset Pricing Model and Abitrage Pricing Theory (1)

EE431/438

Copeland, Thomas E. and J. Fred Weston, Financial Theory and Corporate Policy (4th ed), Addison-Wesley, 2005: Ch6 (pp 147 -157)
Federic Mishkin, The Economics of Money, Banking and Financial Markets (Appendix to Chapter 5, available in the internet):

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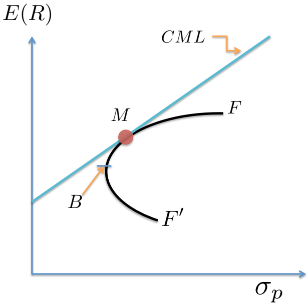
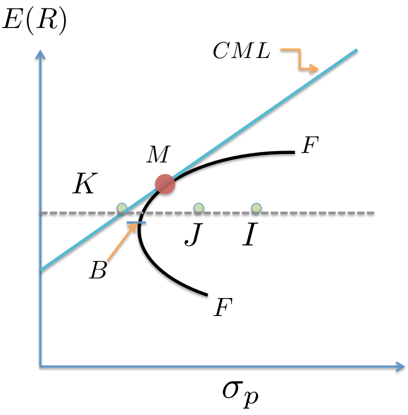
- Risk ⇒ Required Rate of Return ⇒ Price
  $\uparrow$ high $\uparrow$ high $\uparrow$ low

$$Price = \sum_{t} \frac{CF_t}{(1+k)^t}$$

- M-V analysis : Jame Tobin's investment decision process (Separation Theorem)



Portfolio Diversification and Individual Asset Risk



- Should “variance” be a good measurement of “risk” ?
- Asset *I*, *J* are .... the capital market line
- People do not hold asset *I*, *J* separately
- $E(R), \sigma^2$ of asset *I*, *J* $\nrightarrow$ rate of returns the market will require from asset *I*, *J*
- Riskiness of asset *I*, *J* $\rightarrow$ their constribution in the asset portfolio (covariance risk)
- Equilibrium: CAPM , APT

- Should “variance” be a good measurement of “risk”?
- People do not hold any single asset separately.
- For example : a share in a umbrella company. The share price varies depending on the weather condition (weather risk)
  - very high $\sigma \Rightarrow$ very high required rate of return
  - people do not hold a share in an umbrella company separately. They hold a market portfolio, which includes every asset in the economy.
  - They hold shares in a sunblock manufacturer also.

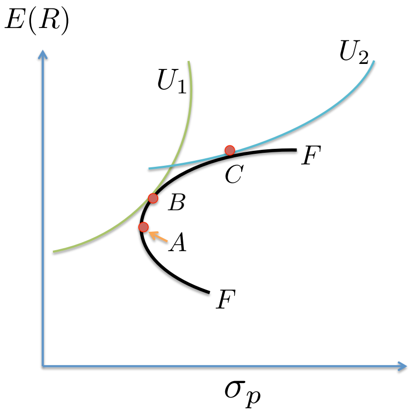
- Holding both shares in a sunblock company and shares in an umbrella company $\Rightarrow \sigma_p \downarrow$
- The porfolio is doing well in all kind of weather (sunny, rainy)
- Therefore, even though the share of an umbrella company may have very high variance. They do not need compensation for weather risk.
- As a result, people will not require very high rate of return to hold hold a share of an umbrella company.
- People consider how individual asset contributes in their asset portfolio (covariance risk).
- $\sigma_m^2 = \sum_{i=1} w_i^2 \sigma_i^2 + \underbrace{\sum_{\substack{i=1 \\ i \neq j}} \sum_{i=1} w_i w_j \sigma_{ij}}_{\text{covariance risk}}$

- CML : $E(R_p) = R_f + \left(\frac{ER_m - R_f}{\sigma_m} \right) \sigma_p$

  Equilibrium relationship between $E(R_p)$ and σ_p for efficiently diversified portfolio (M, R_f)
  NOT Equilibrium relationship between required rate of return and risk for any other asset portfolio.

1. Investors are risk-averse, maximise expected utility of their wealth
2. Investors are price-takers, homogeneous expectations about expected asset returns, variance and covariance
3. There exists a risk-free asset such that investors may borrow or lend unlimitted amount at a risk-free rate
4. The quantities of assets are fixed. All assets are marketable and perfectly divisable
5. Asset market are frictionless, no information cost, no transaction cost
6. no market imperfections such as taxes, regulations, or restrictions on short selling

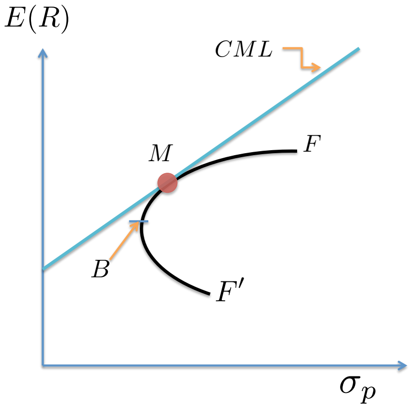
The CAPM : The Efficiency of the Market Porfolio



- Homogeneouse beliefs $\rightarrow$ same efficient frontier
- Even without a risk-free asset, individuals will choose efficient portfolios regardless of their degree of risk aversion
- Market portfolio $\rightarrow$ consisting of all assets, held in proportion equals to its share in the market)
- The market is the sum of all individual holdings

- The market portfolio must lie on the efficient frontier

Derivation of the CAPM (1)



- Homogeneouse beliefs $\rightarrow$ same efficient frontier, same CML
- In equilibrium: Demand = Supply
- The prices must adjust until all assets are held
- All individuals hold a combination of a risk-free asset and the portfolio *M*
- *M* must be a market portfolio

Derivation of the CAPM (2)

- At equilibrium, the proportion of each asset in the market portfolio must be

$$w_i = \frac{\text{market value of the individual asset } i}{\text{market value of ....}}$$

- Consider a portfolio consisiting of a % invested in risky asset i and $(1 - a)\%$ in the market portfolio (a will be equal to 0 later)

$E(R_p) = ...$

$\sigma(R_p) =$...

where
σ_i^2 =the variance of the risky asset i
σ_m^2 =the variance of the market portfolio
σ_{im} =the covariance between the risky asset i and the market portfolio

Derivation of the CAPM (3)

- $\dfrac{\partial E(R_p)}{\partial a} =$
- $\dfrac{\partial \sigma(R_p)}{\partial a} =$

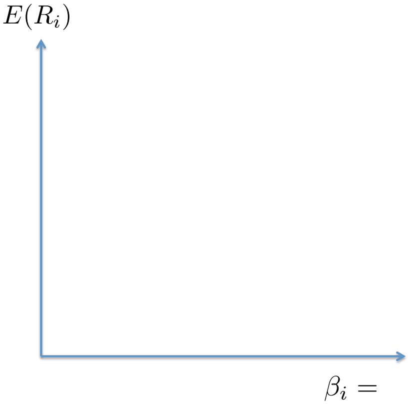
- In equilibrium, $a = 0$. The slope of the efficient portfolio at point M : $\left.\dfrac{\partial E(R_p)/\partial a}{\partial \sigma_p/\partial a}\right|_{a=0}$
- $\dfrac{\partial \sigma(R_p)}{\partial a} = \dfrac{\sigma_{im} - \sigma_m^2}{\sigma_m}$
- $\dfrac{\partial E(R_p)/\partial a}{\partial \sigma_p/\partial a} =$

- Recall CML line : $E(R_p) = R_f + \frac{E(R_p) - R_f}{\sigma_p}$
- The tangency portfolio is the market portfolio; the slopes must be equal
- Rearrange, solve for $E(R_i)$

This equation is known as *the capital asset pricing model*, CAPM.

- It shows the relationship between the required rare of return on any asset and the quantity of risk (β)
- $\beta_i = \frac{\sigma_{im}}{\sigma_m^2} =$
- $\beta_m =$

- The CAPM can be shown graphically.



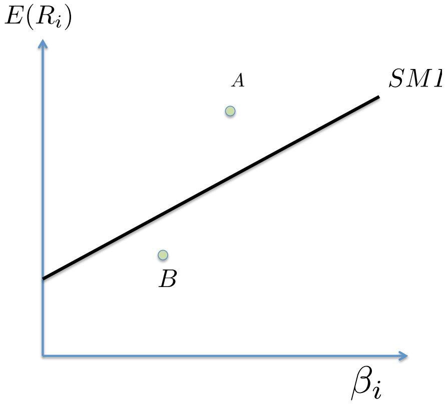
- *Security Market Line* (SML)
- CML : the relationship between expected return and “total” risk for efficiently diversified portfolios
- SML : “equilibrium relationship” between expected return and β - “*Systematic Risk*” (explain later)

Properties of the CAPM (1)

(1) The only risk that investors will pay a premium to avoid is covariance risk ; $\beta_i = \frac{COV(R_i, R_m)}{VAR(R_m)}$

- Investors can always diversify away all risk except the covariance risk
- The total risk = systematic risk(aggregate risk) + unsystematic risk (ideosyncratic risk)
- aggregate risk cannot be diversified away
- Mathematically; $R_i = a_i + b_i R_m + \epsilon_i$
- ϵ_i is a random variable; $E(\epsilon_i) = 0,\ COV(\epsilon_i, \epsilon_j) = 0,\ COV(\epsilon_i, R_m) = 0$
- $\sigma_i^2 =$

(2) In equilibrium, every asset must be priced so that its required rate of return falls on SML



- Asset A is .... (undervalued/overvalued?)
  - investors would .... (buy or sell ? more A)
  - then, the price of asset A will .... (rise/fall?)
  - its required rate of return will .... (rise/fall?)
- Asset B is ....
  - investors would ....
  - then, the price of asset B will ....
  - its required rate of return will ....

$$\textit{Price} = \frac{\textit{Payoff}}{1 + \text{required rate of return}}$$

CAPM summary

- CML : $E(R_p) = R_f + \left(\frac{E(R_m) - R_f}{\sigma_m}\right)\sigma_p$
- CML shows the relationship between expected return and “total” risk for efficiently diversified portfolios. Investors hold portfolios along the CML. Their market required rate of returns are determined by the relationship between the market required rate of return and standard deviation, as shown by CML equation.
- Portfolios which do not lie on the CML are inefficient. Though we know the variance and the rate of return on those portfolios, we cannot be sure at what rate of return that the market will require from them in order to hold them in equilibrium. The variance may not be the correct measure of riskiness for an individual asset.

- Hence, we cannot use our knowledge of the mean and the variance of an individual asset to determine its market required rate of return.
- We derive the CAPM by equating the slope of CML to the slope of efficient frontier at point M.
- SML : $E(R_i) = R_f + ((E(R_m) - R_f)\,\beta_i$; $\beta_i = \frac{\sigma_{im}}{\sigma_m^2} = \frac{COV(R_m, R_i)}{VAR(R_m)}$
- SML : "equilibrium relationship" between expected return and β - "*Systematic Risk*".
- Only the portion of total variance that is correlated with the economy(covariance risk) is relevant. Any portion of total risk that is not correlated with the economy is irrelevant and can be avoided at zero cost through diversification. We need the information on how the rate of return on an asset is correlated with the economy(covariance risk, systematic risk, β) to determine the market required rate of return on an asset.

- R_i is a random variable. $E(R_i)$ is a constant.
- R_m is a random variable. $E(R_m)$ is a constant.
- R_f is a contant.
- σ_i , σ_m, σ_{im} are a constant.
- According to the CAPM, $R_i = R_f + \beta_i(R_m - R_f) + \epsilon_i$.
- $E(R_i) = a_i + \beta_i E(R_m)$; $a_i = R_f - \beta_i R_f$ and $E(\epsilon_i) = 0$
- a_i is a constant.
- $R_i = a_i + \beta_i R_m + \epsilon_i$
- $\sigma_i^2 = \beta_i^2 \sigma_{im}^2 + \sigma_{\epsilon_i}^2$
- Total risk = systematic risk + unsystematic risk.
- All assets will fall on SML in equilibrium.
- $\beta_p = a\beta_X + b\beta_Y$

- The CAPM applies when:
  1. market is in the equilibrium
  2. mean-variance portfolio objective
  3. homogenous expectations
- The CAPM predicts that all investors hold the same risky asset portfolio
- Expected required rate of return on asset can be predicted form the SML
- Total risk = systematic risk + unsystematic risk
- The only risk that investors will pay a premium to avoid is systematic risk
- The CAPM : $E(R_i) = R_f + \beta_i(R_m - R_f)$

- Arbitrage : an abitrage opportunity arises when an investor can construst a zero investment portfolio that will yield a sure profit (risk-free)
- Example :
-

	S_1	S_2	*Price*
A_1	1	0	0.2
A_2	0	1	0.1

- If there exists a security C pay 2 $ when S_1 occurs and it is priced at $ 0.5, we can contruct an abitrage portfolio; selling at a high price and buying at a low price.
- In equilibrium, no abitrage portfolio exists.
- Law of one price

The Abitrage Pricing Theory

- Arbitrage : an abitrage opportunity arises when an investor can construct a zero investment portfolio that will yield a sure profit (risk-free)
- The CAPM : $R_i = R_f + \beta_i(R_m - R_f) + \epsilon_i$: only one source of systematic risk, systematic risk can be eliminated through diversification
- The APT: several sources of risk that cannot be eliminated through diversification
- The sources of risk : inflation, aggregate output, .. etc.
- $R_i = \beta_0^i + \beta_1^i(\text{Factor 1}) + \beta_2^i(\text{Factor 2}) + ... + \beta_k^i(\text{Factor k}) + \epsilon^i$
- If there is only one factor and that factor is R_m, the APT is the same as the CAPM.
- the APT is more general than the CAPM

- CAPM: $E(R_i) = a_i + \beta_i(E(R_m) - R_f))$;
  $E(R_i) = \text{constant}_1 + \beta_i(\text{constant}_2) = a_i + \beta_i E(R_m)$
- APT :
  $E(R_i) = \beta^i_0 + \beta^i_1(\text{Factor 1}) + \beta^i_2(\text{Factor 2}) + ... + \beta^i_k(\text{Factor k})$
- APT : $R_i = \beta^i_0 + \beta^i_1 F_1 + \beta^i_2 F_2 + ... + \beta^i_k F_K$
- Assets with the same values of β_j for all factors j must have the same rate of returns.

- Example : $E(R_a) = 0.08 + 0.6F_1$, $E(R_b) = 0.02 - 0.2F_1$.
  - Determine the portfolio weights you need to place on a and b in order to construct a portfolio which has no risk ($\beta_1 = ...$).
  - If the risk free rate is equal to 0.01 (you can lend or borrow at 1% interest rate), can you make an arbitrage profit?

- At equilibrium, there must be no abitrage opportunity.
- $E(R_i) = \beta_0^i + \beta_1^i F_1 + \beta_2^i F_2 + ... + \beta_k^i F_K$
- Assets with the same values of β_j for all factors j must have the same rate of returns.
- Example : $E(R_a) = 0.08 + 0.6F_1$, $E(R_b) = 0.02 - 0.2F_1$.
  - Determine the portfolio weights you need to place on a and b in order to construct a portfolio which has $\beta_1 = 1$.
  - Determine the rate of returns of portiolio i which has 0, 0.6, 0.2, 1, 0.5, 2.

- $E(R_i) = R_f + (E(R_m) - R_f)\beta_1^i$; R_m = the rate of return of the portfolio which has $\beta_1 = 1$, R_f = the rate of return of the portfolio which has $\beta_1 = 0$.
- Short selling: is the practice of selling asset that have been borrowed from a broker with the intention of buying the same asset back at a later date to return the broker. (This technique is used by investors who try to profit from the falling price of a stock.)
- APT :
  $E(R_i) = \beta_0^i + \beta_1^i(\text{Factor 1}) + \beta_2^i(\text{Factor 2}) + ... + \beta_k^i(\text{Factor k})$
- $E(R_i) = R_f + (\lambda_1 - R_f)\beta_1^i + (\lambda_2 - R_f)\beta_2^i + ... + (\lambda_k - R_f)\beta_k^i$;
- λ_j = the expected rate of return on portfolio with unit sensitivity to the *j* the factor and zero sensitivity to all other factors.
- At equilibrium, there is no arbitrage opportunity.

- Examples: 2 Factors
  $R_a = 0.10F_1 - 0.5F_2$
  $R_b = 0.08 + 2F_1 + F_2$
  $R_c = 0.05 + 0.5F_1 + 0.5F_2$
  $R_p = w_aR_a + w_bR_b + w_cR_c$
- Determine the portfolio weights you need to place on a and b in order to construct a portfolio which has no risk
  $0.10w_a + 2w_b + 0.5w_c = 0$
  $-0.5w_a + 1w_b + 0.5w_c = 0$
  $w_a = \frac{5}{13},\ w_b = -\frac{3}{13},\ w_c = \frac{11}{13}$
- Determine the rate of return on portfolio which has $\beta_1 = 1$ and $\beta_2 = 0$.
- Determine the rate of return on portfolio which has $\beta_1 = 0$ and $\beta_2 = 1$
- $E(R_i) = R_f + (\lambda_1 - R_f)\beta_1^i + (\lambda_2 - R_f)\beta_2^i$; λ_j = the expected rate of return on portfolio which has $\beta_j = 1$ and other betas equal to zero.