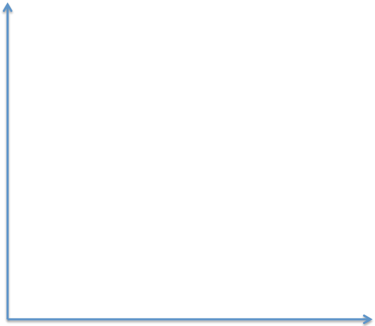
Topic 4 : Capital Asset Pricing Model and Abitrage Pricing Theory

EE431/438

Copeland, Thomas E. and J. Fred Weston, Financial Theory and Corporate Policy (4th ed), Addison-Wesley, 2005: Ch6 (pp 147 -157)
Federic Mishkin, The Economics of Money, Banking and Financial Markets (Appendix to Chapter 5, available in the internet):

September 2012

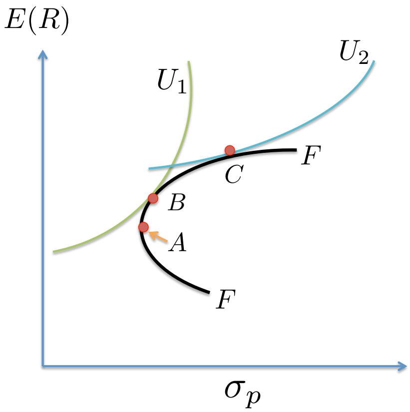
Portfolio Diversification and Individual Asset Risk



- Should "variance" be a good measurement of "risk" ?
- Asset *I*, *J* are .... the capital market line
- People do not hold asset *I*, *J* separately
- $E(R), \sigma^2$ of asset *I*, *J* $\nrightarrow$ rate of returns the market will require from asset *I*, *J*
- Riskiness of asset *I*, *J* $\rightarrow$ their constribution in the asset portfolio (covariance risk)
- Equilibrium: CAPM , APT

The CAPM : Assumptions

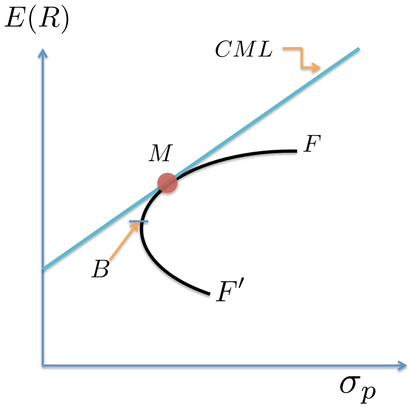
1. Investors are risk-averse, maximise expected utility of their wealth
2. Investors are price-takers, homogeneous expectations about expected asset returns, variance and covariance
3. There exists a risk-free asset such that investors may borrow or lend unlimitted amount at a risk-free rate
4. The quantities of assets are fixed. All assets are marketable and perfectly divisable
5. Asset market are frictionless, no information cost, no transaction cost
6. no market imperfections such as taxes, regulations, or restrictions on short selling



- Homogeneouse beliefs $\rightarrow$ same efficient frontier
- Even without a risk-free asset, individuals will choose efficient portfolios regardless of their degree of risk aversion
- Market portfolio $\rightarrow$ consisting of all assets, held in proportion equals to its share in the market)
- The market is the sum of all individual holdings

- The market portfolio must lie on the efficient frontier

Derivation of the CAPM (1)



- Homogeneouse beliefs $\rightarrow$ same efficient frontier, same CML
- In equilibrium: Demand = Supply
- The prices must adjust until all assets are held
- All individuals hold a combination of a risk-free asset and the portfolio *M*
- *M* must be a market portfolio

- At equilibrium, the proportion of each asset in the market portfolio must be

$$w_i = \frac{\text{market value of the individual asset } i}{\text{market value of ....}}$$

- Consider a portfolio consisiting of a % invested in risky asset i and $(1 - a)\%$ in the market portfolio (a will be equal to 0 later)

$E(R_p) = ...$

$\sigma(R_p) =$...

where
$\sigma_i^2 =$the variance of the risky asset i
$\sigma_m^2 =$the variance of the market portfolio
$\sigma_{im} =$the covariance between the risky asset i and the market portfolio

Derivation of the CAPM (3)

- $\dfrac{\partial E(R_p)}{\partial a} =$
- $\dfrac{\partial \sigma(R_p)}{\partial a} =$

- In equilibrium, $a = 0$. The slope of the efficient portfolio at point M : $\left.\dfrac{\partial E(R_p)/\partial a}{\partial \sigma_p/\partial a}\right|_{a=0}$
- $\dfrac{\partial \sigma(R_p)}{\partial a} = \dfrac{\sigma_{im} - \sigma_m^2}{\sigma_m}$
- $\dfrac{\partial E(R_p)/\partial a}{\partial \sigma_p/\partial a} =$

- Recall CML line : $E(R_p) = R_f + \frac{E(R_p) - R_f}{\sigma_p}$
- The tangency portfolio is the market portfolio; the slopes must be equal

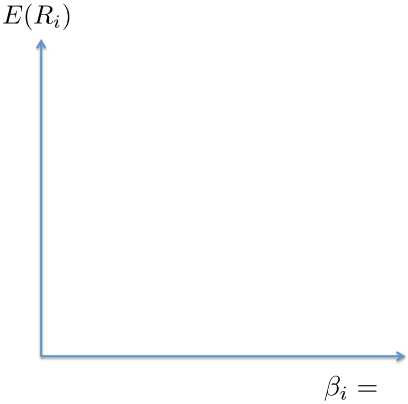
- Rearrange, solve for $E(R_i)$

This equation is known as *the capital asset pricing model*, CAPM.

Derivation of the CAPM (5)

- It shows the relationship between the required rare of return on any asset and the quantity of risk (β)
- $\beta_i = \frac{\sigma_{im}}{\sigma_m^2} =$
- $\beta_m =$

- The CAPM can be shown graphically.



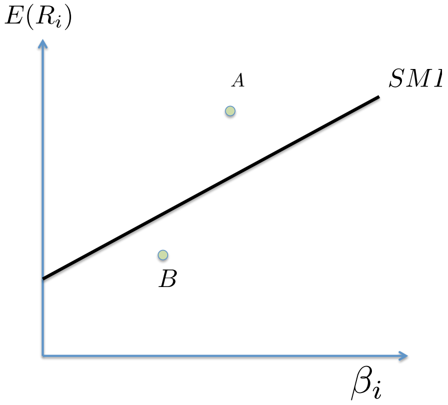
- *Security Market Line* (SML)
- CML : the relationship between expected return and "total" risk for efficiently diversified portfolios
- SML : "equilibrium relationship" between expected return and β - "*Systematic Risk*" (explain later)

Properties of the CAPM (1)

(1) The only risk that investors will pay a premium to avoid is covariance risk ; $\beta_i = \dfrac{COV(R_i, R_m)}{VAR(R_m)}$

- Investors can always diversify away all risk except the covariance risk
- The total risk = systematic risk(aggregate risk) + unsystematic risk (ideosyncratic risk)
- aggregate risk cannot be diversified away
- Mathematically; $R_i = a_i + b_i R_m + \epsilon_i$
- ϵ_i is a random variable; $E(\epsilon_i) = 0,\ COV(\epsilon_i, \epsilon_j) = 0,\ COV(\epsilon_i, R_m) = 0$
- $\sigma_i^2 =$

(2) In equilibrium, every asset must be priced so that its required rate of return falls on SML



- Asset A is .... (undervalued/overvalued?)
  - investors would .... (buy or sell ? more A)
  - then, the price of asset A will .... (rise/fall?)
  - its required rate of return will .... (rise/fall?)
- Asset B is ....
  - investors would ....
  - then, the price of asset B will ....
  - its required rate of return will ....

$$\textit{Price} = \frac{\textit{Payoff}}{1 + \text{required rate of return}}$$

The CAPM: Summary

- The CAPM applies when:
  1. market is in the equilibrium
  2. mean-variance portfolio objective
  3. homogenous expectations
- The CAPM predicts that all investors hold the same risky asset portfolio
- Expected required rate of return on asset can be predicted form the SML
- Total risk = systematic risk + unsystematic risk
- The only risk that investors will pay a premium to avoid is systematic risk
- The CAPM : $E(R_i) = R_f + \beta_i(R_m - R_f)$

The Abitrage Pricing Theory : abitrage opportunity

- Arbitrage : an abitrage opportunity arises when an investor can construst a zero investment portfolio that will yield a sure profit (risk-free)
- Example :
-

	S_1	S_2	*Price*
A_1	1	0	0.2
A_2	0	1	0.1

- If there exists a security C pay 2 $ when S_1 occurs and it is priced at $ 0.5, we can contruct an abitrage portfolio; selling at a high price and buying at a low price.
- In equilibrium, no abitrage portfolio exists.
- Law of one price

The Abitrage Pricing Theory

- the CAPM : $R_i = R_f + \beta_i(R_m - R_f) + \epsilon_i$: only one source of systematic risk, systematic risk can be eliminated through diversification
- the APT: several sources of risk that cannot be eliminated through diversification
- The sources of risk : inflation, aggregate output, .. etc.
- $R_i = \beta^i_0 + \beta^i_1(\text{Factor 1}) + \beta^i_2(\text{Factor 2}) + ... + \beta^i_k(\text{Factor k}) + \epsilon^i$
- the APT is more general than the CAPM

- Example : $R_a = 0.08 + 0.6F_1$, $R_b = 0.02 - 0.2F_1$.
    - Determine the portfolio weights you need to place on a and b in order to construct a portfolio which has no risk ($\beta_1 = ...$).
    - If the risk free rate is equal to 0.01 (you can lend or borrow at 1% interest rate), can you make an arbitrage profit?

- At equilibrium, there must be no abitrage opportunity.
- $R_i = \beta_0^i + \beta_1^i F_1 + \beta_2^i F_2 + ... + \beta_k^i F_K$
- Assets with the same values of β_j for all factors j must have the same rate of returns.
- Example : $R_a = 0.08 + 0.6F_1$, $R_b = 0.02 - 0.2F_1$.
  - Determine the portfolio weights you need to place on a and b in order to construct a portfolio which has $\beta_1 = 1$.
  - Determine the rate of returns of portiolio i which has 0, 0.6, 0.2, 1, 0.5, 2.

- Examples: 2 Factors
  $R_a = 0.10F_1 - 0.5F_2$
  $R_b = 0.08 + 2F_1 + F_2$
  $R_c = 0.05 + 0.5F_1 + 0.5F_2$
  $R_p = w_a R_a + w_b R_b + w_c R_c$
- Determine the portfolio weights you need to place on a and b in order to construct a portfolio which has no risk
  $0.10w_a + 2w_b + 0.5w_c = 0$
  $-0.5w_a + 1w_b + 0.5w_c = 0$
  $w_a = \frac{5}{13},\ w_b = -\frac{3}{13},\ w_c = \frac{11}{13}$