Q1. Given the production function $Q = f(K, L) = A[K^n + L^n]$ where A is the level of technology, K is capital and L is labor. Suppose that $n > 0$. Consider the following problem.

a. To ensure that the above production function exhibits a *decreasing return to scale technology*, what additional restrictions do one need to place on n?

b. Under the assumption used in (a), show that the production function satisfies the law of diminishing returns.

c. Calculate the marginal rate of technical substitution (MRTS) of labor (L) for capital (K).

d. Show that MRTS is a decreasing function in L. That is, as labor increases, the value of MRTS decreases.

e. Under the condition(s) assumed in (a), is the production function satisfied with the *global* concave property?

a) $Q = A[K^n + L^n]$

$\Rightarrow A[(tK)^n + (tL)^n]$

$= At^n[K^n + L^n]$

$= t^n(Q)$

Decreasing return to scale ; $Q\,f(K,L) = A[K^\alpha \cdot L^\beta]$

when $\alpha + \beta < 1$

then $n + n < 1$

$n < \frac{1}{2}$

$0 < n < \frac{1}{2}$

b)

MPK ; $\frac{\partial Q}{\partial K} = nAK^{n-1}$

$\frac{\partial MPK}{\partial K} = n(n-1)AK^{n-2} < 0$ — K increases, MPK falls due to law of diminishing returns

from a) $0 < n < \frac{1}{2}$

$n - 1 < \frac{1}{2}$

MPL ; $\frac{\partial Q}{\partial L} = nAL^{n-1}$

$\frac{\partial MPL}{\partial L} = n(n-1)AL^{n-2} < 0$ — L increases, MPL falls due to law of diminishing returns

c) $MRTS = -\frac{F_L}{F_K} = -\frac{nAL^{n-1}}{nAK^{n-1}} = -\left(\frac{L}{K}\right)^{n-1}$ #

d) $\frac{dMRTS}{dL} = -(n-1)\frac{L^{n-2}}{K^{n-1}}$

when L increases by 1 unit, MRTS decreases by $(n-1)\frac{L^{n-2}}{K^{n-1}}$ units

e) S.O.C.

$$H = \begin{bmatrix} F_{KK} & F_{KL} \\ F_{LK} & F_{LL} \end{bmatrix} = \begin{bmatrix} n(n-1)AK^{n-2} & 0 \\ 0 & n(n-1)AL^{n-2} \end{bmatrix}$$

$|H_1| = n(n-1)AK^{n-2} < 0$

$|H_2| = n^2(n-1)^2A^2K^{n-2}L^{n-2} > 0$

alternate sign

negative definite for $\forall_K, \forall_L$

Q is globally concave

Suppose that $K(t) = \frac{1}{2}t^2 + 2t + 3$ and $L(t) = e^t + 3$, where $t \geq 0$ is the number of periods from now. Consider the following problem

f. Show that Q is increasing over time.

g. Compute $\frac{\frac{dQ}{Q}}{dt}$ when t = 0, i.e. growth of output in the initial period.

f) $\frac{\partial Q}{\partial t} = \frac{\partial Q}{\partial K}\cdot\frac{\partial K}{\partial t} + \frac{\partial Q}{\partial L}\cdot\frac{\partial L}{\partial t}$

$= \underbrace{(nAK^{n-1})(t+2)}_{+} + \underbrace{(nAL^{n-1})(e^t)}_{+}$

when t increases ; t+2 increases; Q increase

e > 1 as t increasing, e^t is also increasing ; Q increase

To conclude, when t rises, Q also rises

g) $t = 0$; $K = \frac{1}{2}(0^2) + 2(0) + 3 = 3$

$L = e^0 + 3 = 4$

$Q = A[K^n + L^n] = A[3^n + 4^n]$

$\left.\frac{\partial Q}{\partial t}\right|_{t=0} = nA3^{n-1}(0+2) + nA4^{n-1}(e^0)$

$= nA(2\cdot 3^{n-1} + 4^{n-1})$

$\frac{\frac{dQ}{Q}}{dt} = \frac{dQ}{dt}\cdot\frac{1}{Q} = nA(2\cdot 3^{n-1} + 4^{n-1})\cdot\frac{1}{A(3^n+4^n)}$

$= \frac{n(2\cdot 3^{n-1} + 4^{n-1})}{3^n + 4^n}$ #

which is positive

Q2. A monopolist faces the market demand given by $P = Q^{-c}$ where "c" is a parameter with positive value, "P" is the price per unit output and "Q" is the amount of output. Suppose that monopolist's production technology is given by $Q = K^{\frac{1}{3}}L^{\frac{2}{3}}$ where "K" and "L" are the level of capital used and the number of labor employed, respectively. Assume that the unit price of K and L are set equal to "r" and "w", respectively. Consider the following problems.

a. What type of the return to scale technology does the production function exhibit?

From now on, assume that c = ¼ . Consider the following problems.

b. Construct the profit function of the monopolist. (Hint: your profit function should be expressed in terms of K and L.)

c. The firm wants to maximize profit and seek for combination of the two factor inputs . Derive the demand for factor inputs, capital and labor.

d. How does the demand for labor vary with respect to w and r? Show your result by using partial derivative.

e. Confirm your answer with the second-order condition.

a) $Q = K^{\alpha} \cdot L^{\beta} = K^{1/3} L^{2/3}$

$\alpha + \beta = \frac{1}{3} + \frac{2}{3} = 1 \Rightarrow$ constant return to scale #

b) $\pi = TR - TC$

$= Q^{-c} \cdot Q - wL - rK$

$= (K^{1/3} L^{2/3})^{1-\frac{1}{4}} - wL - rK$

$\pi = K^{1/4} L^{1/2} - wL - rK$ #

c) F.O.C.

$\frac{\partial \pi}{\partial K} = \frac{1}{4} K^{-3/4} L^{1/2} - r = 0$

$K^* = \left(\frac{4r}{L^{1/2}}\right)^{-4/3}$

$K^* = \frac{L^{2/3}}{(4r)^{4/3}}$

$\frac{\partial \pi}{\partial L} = \frac{1}{2} K^{1/4} L^{-1/2} - w = 0$

$L^* = \left(\frac{2w}{K^{1/4}}\right)^{-2}$

$L^* = \frac{K^{1/2}}{4w^2}$

$K^* = \frac{L^{2/3}}{(4r)^{4/3}}$; $L^* = \frac{K^{1/2}}{4w^2}$

$K = \frac{\left(\frac{K^{1/2}}{4w^2}\right)^{2/3}}{(4r)^{4/3}}$

$K = \frac{K^{1/3}}{4^{2/3} w^{4/3} \cdot 4^{4/3} r^{4/3}}$

$K^{2/3} = \frac{1}{16 w^{4/3} r^{4/3}}$

$K^* = \frac{1}{64 w^2 r^2}$ #

capital demand function

$L^* = \frac{K^{1/2}}{4w^2}$

$= \frac{\left(\frac{1}{64 w^2 r^2}\right)^{1/2}}{4w^2}$

$= \frac{1}{8wr \cdot 4w^2}$

$L^* = \frac{1}{32 w^3 r}$ #

labor demand function

$Q(K^*, L^*) = (K^*)^{1/3} (L^*)^{2/3}$

$= \frac{1}{4 w^{2/3} r^{2/3}} \cdot \frac{1}{4^{5/3} w^2 r^{2/3}}$

$= \frac{1}{4^{8/3} w^{8/3} r^{4/3}}$ #

d) $L^* = \dfrac{1}{32 w^3 r} = \dfrac{1}{32} w^{-3} r^{-1}$

$\dfrac{\partial L^*}{\partial w} = \dfrac{-3}{32} w^{-4} r^{-1}$

when unit price of labor increases by 1 unit, demand for labor decreases by $\dfrac{3}{32} w^{-4} r^{-1}$ units

$\dfrac{\partial L^*}{\partial r} = \dfrac{-1}{32} w^{-3} r^{-2}$

when unit price of capital increases by 1 unit, demand for labor decrease by $\dfrac{1}{32} w^{-3} r^{-2}$ units

e) S.O.C

$H = \begin{bmatrix} \pi_{KK} & \pi_{KL} \\ \pi_{LK} & \pi_{LL} \end{bmatrix} = \begin{bmatrix} \frac{-3}{16} K^{-7/4} L^{1/2} & \frac{1}{8} K^{-3/4} L^{-1/2} \\ \frac{1}{8} K^{-3/4} L^{-1/2} & \frac{-1}{4} K^{1/4} L^{-3/2} \end{bmatrix}$

$|H_1| = -\dfrac{3}{16} K^{-7/4} L^{1/2}$ <0

$|H_2| = \left(\frac{-3}{16} K^{-7/4} L^{1/2}\right)\left(\frac{-1}{4} K^{1/4} L^{-3/2}\right) - \left(\frac{1}{8} K^{-3/4} L^{-1/2}\right)^2$ >0

alternate sign $\Rightarrow$ negative definite for $\forall_K, \forall_L$
π is globally concave
(K^*, L^*) from F.O.C. is true global max solution

a)

firm 1

$\pi^1 = p \cdot q_1 - c_1$

$= [100 - (q_1+q_2)] \cdot q_1 - 10q_1$

$\frac{\partial \pi^1}{\partial q_1} = 100 - 2q_1 - q_2 - 10 = 0$

$q_2 = 90 - 2q_1$

firm 2

$\pi^2 = p \cdot q_2 - c_2$

$= [100 - (q_1+q_2)]q_2 - q_2^2$

$\frac{\partial \pi^2}{\partial q_2} = 100 - q_1 - 2q_2 - 2q_2 = 0$

$q_1 = 100 - 4q_2$

$q_2 = 90 - 2(100 - 4q_2)$

$110 = 7q_2$

$q_2 = \frac{110}{7}$ units

$q_1 = \frac{260}{7}$ units

$\Big\}\ p = 100 - (q_1 + q_2) = \frac{330}{7}$ #

Q3. In a duopoly market, suppose that firms 1 and 2 face market demand, $p = 100 - (q_1 + q_2)$. Firm costs are $c_1 = 10q_1$ and $c_2 = q_2^2$.

(a) Calculate market price under the Cournot environment. How much is the profit that each firm yields in the equilibrium?

(b) Calculate the market price under the joint production decision, i.e. collusive equilibrium. How much is the profit that each firm yields in the equilibrium?

(c) Do you think that the collusive equilibrium will sustain? What would be the profit of firm 1 if firm 1 chooses to deviate from the collusive allocation?

$\pi^1 = \frac{330}{7} \times \frac{260}{7} - 10\left(\frac{260}{7}\right)$

$= \frac{67600}{49} \approx 1379.59$ #

$\pi^2 = \frac{330}{7} \times \frac{110}{7} - \left(\frac{110}{7}\right)^2$

$= \frac{24200}{49} \approx 493.88$ #

b) $\max \pi = \pi^1 + \pi^2$

$= P \cdot (q_1+q_2) - C_1 - C_2$

$= (100 - (q_1+q_2))(q_1+q_2) - 10q_1 - q_2^2$

$\frac{\partial \pi}{\partial q_1} = 100 - 2(q_1+q_2) - 10 = 0$

$\frac{\partial \pi}{\partial q_2} = 100 - 2(q_1+q_2) - 2q_2 = 0$

$2q_2 = 10$

$q_2 = 5$

$q_1 = 40$

$Q = 45$

$P = 100 - 45 = 55$ #

with collusive $\pi^1 = (55)(40) - 10(40)$

$= 1800$ #

$\pi^2 = (55)(5) - 5^2$

$= 250$ #

c)

	cournot	collusive
price	47.14	55
q_1	37.14	40
q_2	15.71	5
π^1	1379.59	1800
π^2	493.88	250

Compairing between cournot and collusive
firm 1 will get higher price, higher quantity, and higher profit
so, it wants to sustain in collusive equilibrium.
However firm 2 will get higher price, lower quantity,
and lower profit
so, firm 2 might not want to sustain in collusive equilibrium

If firm 1 deviate from collusive allocation, its profit
will fall by 420.41

Q4. Consider a simple two-market model where demand and supply for each market is given below. (Notationally, let's name the two markets as A and B, respectively.)

Market A:

Demand: $p_A = 10 - 2Q_A$

Supply: $p_A = 1 + Q_A$

Market B:

Demand: $p_B = 20 - Q_B$

Supply: $p_B = 2 + 2Q_B$

a. Derive the market equilibrium
b. Suppose the government imposes unit tax on both markets at the rate of t_A and t_B. Solve for the after-tax equilibrium as the function of t_A and t_B.
c. How much revenue can the government collect from the taxation?
d. Determine the level of t_A and t_B that maximizes government's revenue.

a) market A

Demand = Supply

$10 - 2Q_A = 1 + Q_A$

$Q_A^* = 3$ #

$P_A^* = 4$ #

market B

Demand = Supply

$20 - Q_B = 2 + 2Q_B$

$Q_B^* = 6$ #

$P_B^* = 14$ #

b) $p^s = p^d - t$

$p_A^s = p_A^d - t_A$

$1 + Q_A = 10 - 2Q_A - t_A$

$Q_A = \frac{9 - t_A}{3}$

$p_A^s = \frac{12 - t_A}{3}$

$p_A^d = \frac{12 + 2t_A}{3}$

$p^s = p^d - t$

$p_B^s = p_B^d - t_B$

$2 + 2Q_B = 20 - Q_B - t_B$

$Q_B = \frac{18 - t_B}{3}$

$p_B^s = \frac{42 - 2t_B}{3}$

$p_B^d = \frac{42 + t_B}{3}$

c) Revenue from A $= t_A \cdot Q_A$

from B $= t_B \cdot Q_B$

R = total revenue $= t_A\left(\frac{9 - t_A}{3}\right) + t_B\left(\frac{18 - t_B}{3}\right) = 3t_A - \frac{t_A^2}{3} + 6t_B - \frac{t_B^2}{3}$ #

d) F.O.C.

$\frac{\partial R}{\partial t_A} = 3 - \frac{2}{3}t_A = 0$

$t_A = 4.5$ #

$\frac{\partial R}{\partial t_B} = 6 - \frac{2}{3}t_B = 0$

$t_B = 9$ #

Q5. (Monopolist and the Optimal advertising)

Consider a linear demand equation faced by a monopolist.

$$Q = 2000 + 4\sqrt{A} - 20P,$$

where A is the dollar amount of the expenditure on advertisement, P is the unit price, and Q is the amount of quantity demanded by consumers. Both θ and β are strictly positive.

The demand equation is an extended version of the traditional one. The proposed function captures an idea that advertising can boost the total demand in the market as potential customers get more information about the product.

Suppose that the monopolist cost function is given by $C(Q, A) = c(Q) + A$ where $c(Q) = 2Q + 1000$

Consider the following problem.

a) Construct the profit function.
b) Determine the optimal pricing (P*) and optimal advertisement (A*) that maximize the profit of the monopolist.
c) Confirm the result with the second-order differential test, i.e. hessian-matrix method.

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a) $\pi = P \cdot Q - TC$

$= P(2000 + 4\sqrt{A} - 20P) - (2Q + 1000 + A)$

$= P(2000 + 4\sqrt{A} - 20P) - 2(2000 + 4\sqrt{A} - 20P) - 1000 - A$

$\pi = 2040P + 4P\sqrt{A} - 20P^2 - 8\sqrt{A} - 5000 - A$ #

b) $\frac{\partial \pi}{\partial P} = 2040 + 4\sqrt{A} - 40P = 0$

$P = \frac{2040 + 4\sqrt{A}}{40} = \frac{2040 + 4(2P-4)}{40}$

$\frac{\partial \pi}{\partial A} = 2PA^{-1/2} - 4A^{-1/2} - 1 = 0$

$A^{-1/2} = \frac{1}{2P-4}$

$40P = 2040 + 8P - 16$

$32P = 2024$

$P^* = 63.25$ #

$A^{-1/2} = \frac{1}{122.5} \longrightarrow A^{1/2} = 122.5 \longrightarrow A^* = 15{,}006.25$ #

c) $H = \begin{bmatrix} \pi_{PP} & \pi_{PA} \\ \pi_{AP} & \pi_{AA} \end{bmatrix} = \begin{bmatrix} -40 & 2A^{-1/2} \\ 2A^{-1/2} & -PA^{-3/2} + 2A^{-3/2} \end{bmatrix}$

$|H_1| = -40 < 0$

$|H_2| = (-40)[-(63.25)(15{,}006.25)^{-3/2} + 2(15{,}006.25)^{-3/2}] - 4A^{-1} > 0$

alternate sign $\Rightarrow$ negative definite for $\forall P, \forall A$

π is globally concave $\Rightarrow$ ($P^* = 63.25, A^* = 15006.25$) is true global max solution.