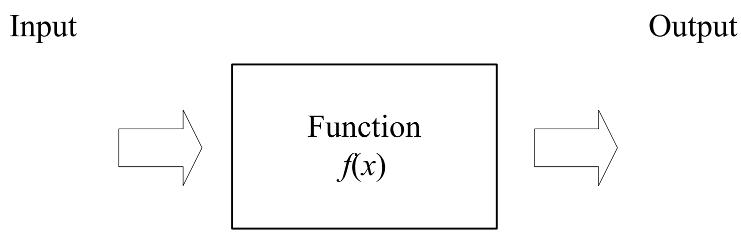
1 Why study calculus?

Calculus is a study of change both in terms of rate of change and total change. Hence the application of calculus in economics can be:

- Optimization: maximize profit, minimize cost, maximize revenue
- Predicting changes of demand and supply when price of a product changes
- Study of marginal changes *i.e.*, incremental changes in work hours or factory output

2 Functions, Graphs and Limits

2.1 Definition of functions



A function is a rule that produces a correspondence between two sets of elements that to each element in the first set (domain) there corresponds **one and only one** element in the second (range) i.e. one-to-one or many-to-one but not one-to-many.

- $y = f(x_1)$ means the value of f at x_1.
- If a domain is not specified, the domain is real number that excluding the numbers that result in range of not-real number. For example x that results in f(x) which is a number dividing by zero or a square root of negative number.

Ex. 1: Determine the following functions for the given values of *x*.

(a) $f(4)$ if $f(x) = x^2 + 4$

Ans: 20

(b) $f(38), f(4), f(2)$ and $f(1.8)$ if $f(t) = (t-2)^{\frac{1}{2}}$

Ans: 6, 1.414, 0, not a real number

(c) $f\left(-\frac{1}{4}\right), f(1)$ and $f(3)$ if $f(x) = \begin{cases} \frac{1}{x-1} & if \quad x < 1 \\ 3x^2 + 1 & if \quad x \geq 1 \end{cases}$

Ans: $-\frac{4}{5}, 4$ and 28

Ex. 2:**Define a domain of the following functions:**

(a) $y = f(x) = 11 - 2x$ Ans: $D \subset \Re$

(b) $g(x) = \frac{7}{5-x}$ Ans: $D \subset \Re - \{5\}$

(c) $h(t) = 7 - 2t^2$ Ans: $D \subset \Re$

(d) $m(t) = \sqrt{3-t}$ Ans: $D = \{t : t < 3\}$

2.1.1 Functions Used in Economics

A ***demand function*** $p = D(x)$ is a function that relates the unit price p for a particular commodity to the number of units, x, demanded by consumers at that price.

A ***total revenue*** is given by the product of number of units, x and price per unit, p

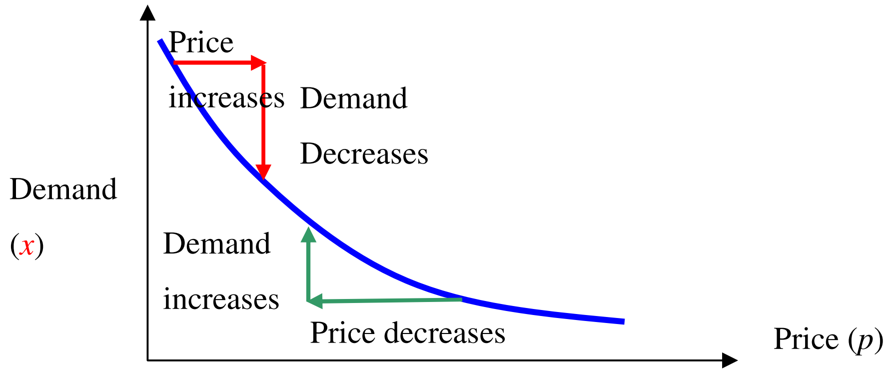
$R(x)$ = (number of items sold)(price per item)

=

If $C(x)$ is the **total cost** of producing the x units, then the **profit** derived from sale at the unit price p is given by the function

$P(x)$ = total revenue – total cost

=



Ex. 3: Market research indicates that consumers will buy x thousand units of a particular kind of mobile phones when the unit price is

$$p = -0.27x + 51 \qquad \text{bahts}$$

The cost of producing the x thousand units is

$$C(x) = 2.23x^2 + 3.5x + 85 \qquad \text{bahts}$$

(a) What are the demand, revenue, and profit function, $D(x)$, $R(x)$, and $P(x)$, for this production process?

(b) For what values of x is production of the mobile phones profitable?

(c) Determine the profit of selling 1, 15 and 20 units.

Ans: (a) $D(x) = -0.27x + 51, R(x) = -0.27x^2 + 51x, P(x) = -2.5x^2 + 47.5x - 85$;
(b) $2 < x < 17$; (c) -40, 65, -135

2.1.2 Composition of Functions

Given function $f(u)$ and $g(x)$, the composition $f(g(x))$ is the function of x formed by substituting $u = g(x)$ for u in the formula for $f(u)$.

For example,

Ex. 4

(a) Find the composite function $f(g(x))$, where $f(u) = u^2 + 3u + 1$ and $g(x) = x + 1$.

(b) Find the composite function $g(f(x))$, where $f(x) = x^2 + 3x + 1$ and $g(t) = t + 1$.

(c) Show that $f(g(x)) = g(f(x))$ only when $x = \frac{3}{2}$.

Ans: $f(g(x)) = x^2 + 5x + 5$; $g(f(x)) = x^2 + 3x + 2$

Ex. 5: Find $f(x-1)$ if $f(x) = 3x^2 + \frac{1}{x} + 5$.

Ans: $f(x-1) = 3(x-1)^2 + \frac{1}{(x-1)} + 5$

Ex. 6: Find **a difference quotient** of $f(x) = x^2 - 3x$ where the difference quotient is defined as $\frac{f(x+h) - f(x)}{h}$. Ans: $2x + h - 3$

Ex. 7: An environmental study of a certain community suggests that the average daily level of carbon monoxide in the air will be $c(p) = 0.5p + 1$ parts per million (ppm) when the population is p thousand. It is estimated that t years from now the population of the community will be $p(t) = 10 + 0.1t^2$ thousand.

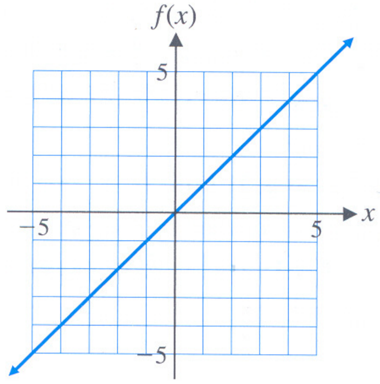
(a) Express the level of carbon monoxide in the air as a function of time.

(b) When will the carbon monoxide level reach 6.8 ppm?

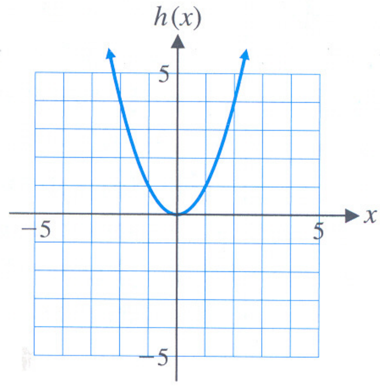
Ans: $c(t) = 6 + 0.05t^2$, 4 years.

2.2 The Graph of a Function

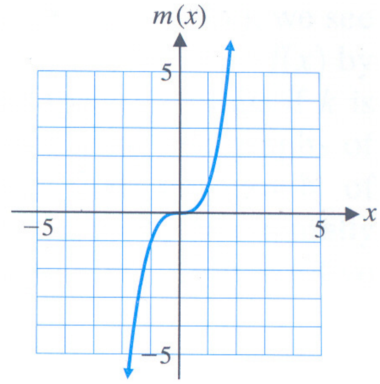
The graph of a function *f* consists of all points (x, y) where *x* is the domain of *f* and $y=f(x)$; that is, all points of the form $(x, f(x))$. The most common coordinate system is Cartesian (rectangular) coordinate system. One of the basic ways to sketch the graph of an equation is to do point by point plotting. Below are 6 basic functions that are used very often.



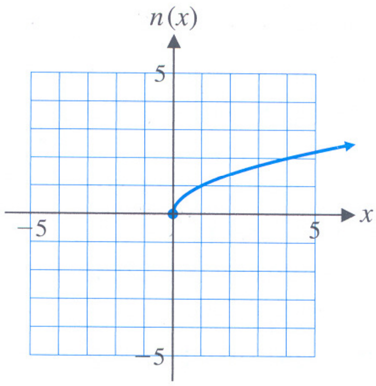
(A) **Identity function**
$f(x) = x$
Domain: R
Range: R



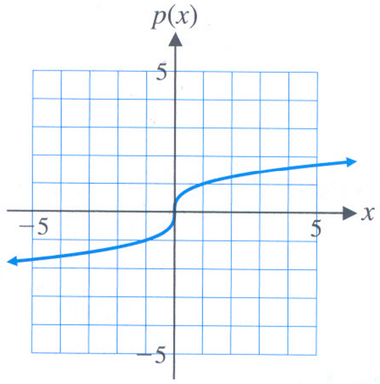
(B) **Square function**
$h(x) = x^2$
Domain: R
Range: $[0, \infty)$



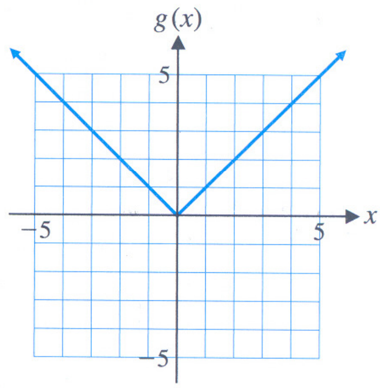
(C) **Cube function**
$m(x) = x^3$
Domain: R
Range: R



(D) **Square root function**
$n(x) = \sqrt{x}$
Domain: $[0, \infty)$
Range: $[0, \infty)$



(E) **Cube root function**
$p(x) = \sqrt[3]{x}$
Domain: R
Range: R



(F) **Absolute value function**
$g(x) = |x|$
Domain: R
Range: $[0, \infty)$

Note: $g(x) = |x| = \begin{cases} x & \text{if} \quad x \geq 0 \\ -x & \text{if} \quad x < 0 \end{cases}$

2.2.1 Intercepts

The points (if any) where a graph crosses the x axis are called ***x* intercepts**, and similarly, a ***y* intercept** is a point where the graph crosses the y axis.

Any x intercept of a graph can be found by setting y = 0 and solving for x. Similarly any y intercept can be found by setting x = 0 and solving for y.

Ex. 8: Find all x and y intercepts of $f(x) = -x^2 + x + 2$.

Ans: (0,2), (-1,0), (2,0)

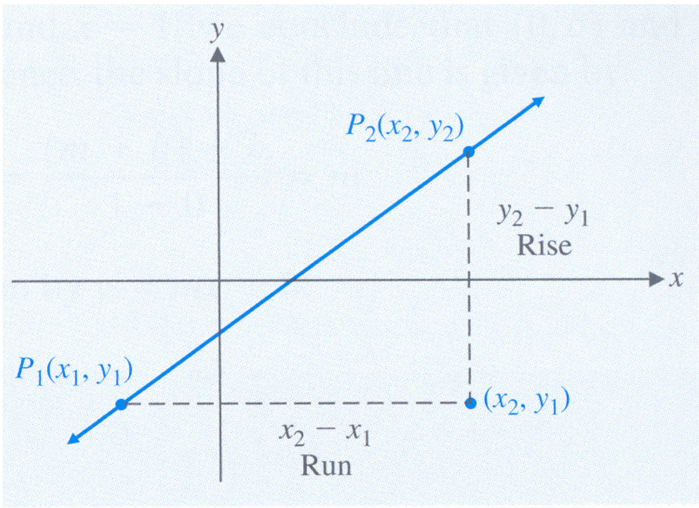
2.2.2 Linear Function and Straight Line

General form:

$y = f(x) = mx + b$ (1st-degree polynomial)

or $y_2 - y_1 = m(x_2 - x_1)$

or $Ax + By = C$



Where $m, b, A, B \in \Re$.

Domain is ______________

Range is _______________

y intercept is ______________

x intercept is ______________

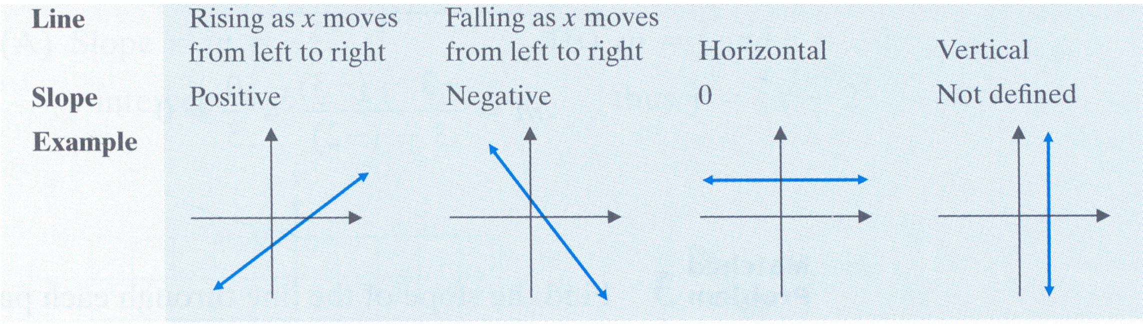
m is the slope of the straight line.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

For two lines with the non-zero slopes of m_1 and m_2,

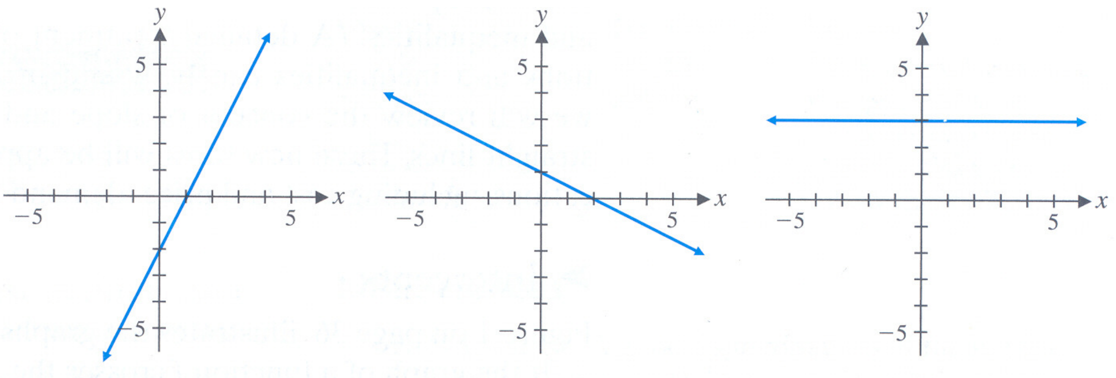
- they are **parallel** if

- they are **perpendicular** if



Line	Rising as x moves from left to right	Falling as x moves from left to right	Horizontal	Vertical
Slope	Positive	Negative	0	Not defined
Example				

Examples graphs of linear functions $f(x) = 2x - 2$, $g(x) = -0.5 + 1$, and $h(x) = 3$.



(A) $f(x) = 2x - 2$ (B) $g(x) = -0.5x + 1$ (C) $h(x) = 3$

Ex. 9: A manufacturer's total cost consists of a fixed overhead of ฿200 plus production costs of ฿50 per unit. Express the total cost as a function of the number of units produced and sketch the graph.

Ans: $C(x) = 50x + 200$

Ex. 10: Find the slope of the line joining the point (-2, 5) and (3, -1).

Ans: -6/5

Ex. 11: Find the slope and x and y intercept of the line $3y + 2x = 6$ and sketch the graph.

Ans: -2/3 and (0,2)

Ex. 12: If a straight line passes a point (5,1) and has slope equal to 0.5. Find the straight line equation.

Ans: $y = 0.5x - 1.5$

Ex. 13: If a straight line passes 2 points (3,-2) and (1,6). Find the straight line equation.

Ans: $y = 10 - 4x$

Ex. 14: Since the beginning of the year, the price of whole wheat bread at a local discounted supermarket has been rising at a constant rate of 2 cents per month. By November first, the price has reached $1.56 per loaf. Express the price of the bread as a function of time (in month) and determine the price at the beginning of the year.

Ans: $1.36

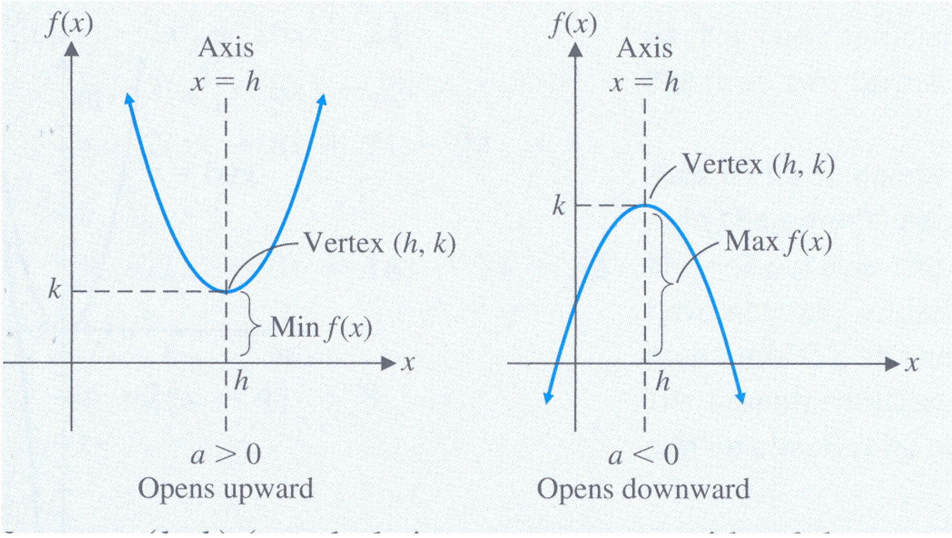
2.2.3 Quadratic Function

General form:

$y = f(x) = ax^2 + bx + c$ (2^{nd}-degree polynomial)

or $y = f(x) = a(x-h)^2 + k$ "Vertex" form

Where $\{a, b, c, h, k\} \in \Re$.



- The graph of $f(x)$ above is parabolic.
- Vertex of the graph is (h,k). Parabola graph increases on one side of the vertex and decreases on the other side.
- Axis of symmetry is $x = h$ (parallel to y axis).
- $f(h) = k$ is the minimum if $a > 0$ and the maximum if $a < 0$.
- Domain is ________________________ and

  Range is ____.
- The graph of $f(x)$ is the graph of $g(x) = ax^2$ translated horizontally h units and vertically k units.

Ex. 15: A manufacturer determines that when x hundred units of a particular commodity are produced, they can all be sold for a unit price given by the demand function. At what level of production is revenue maximised? What is the maximum revenue?

Ans: 30, 900

Ex. 16: Find all points of intersection of graphs of $f(x) = 3x + 2$ and $g(x) = x^2$.

Ans: $x = \frac{3 \pm \sqrt{17}}{2} \approx 3.56 \quad and \quad -0.56$

2.2.4 Polynomial function with n[th] degree

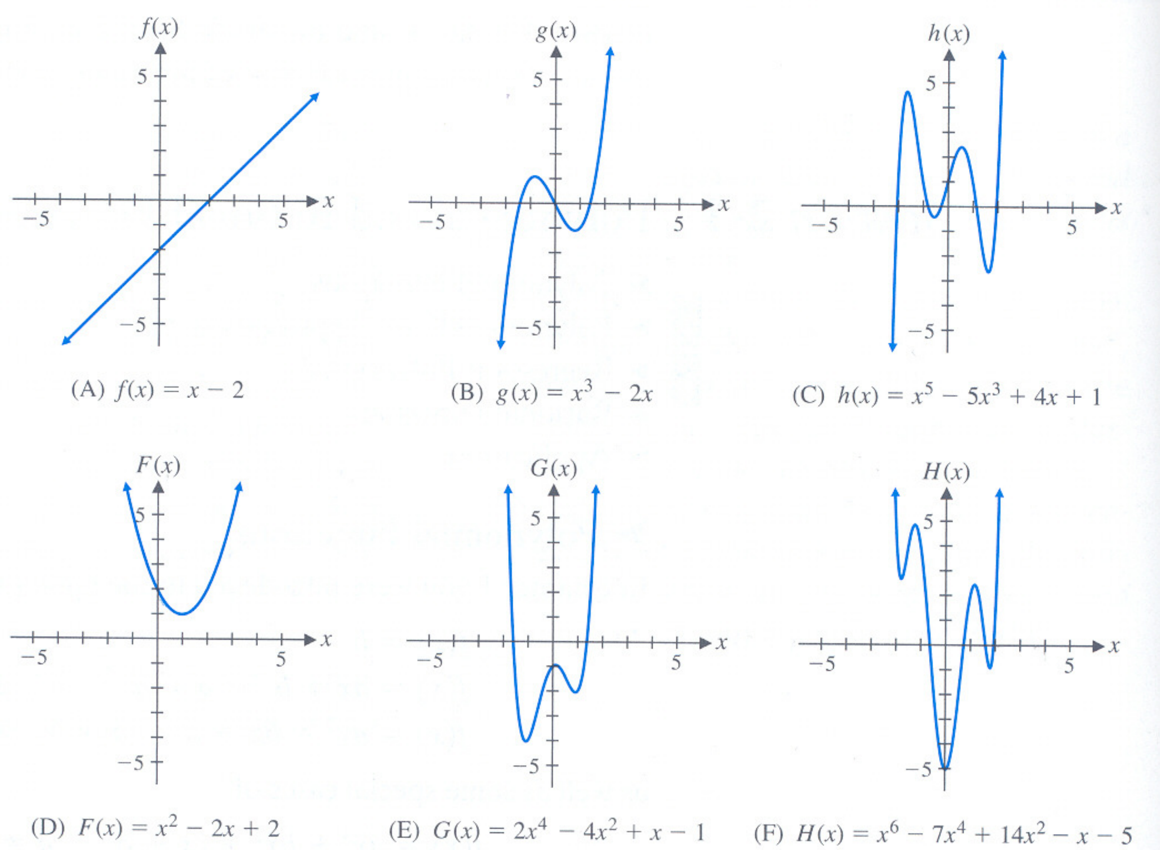
General form:

$$y = f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + ... + a_1 x + a_0$$

For n as a non-negative integer (degree) and $a_0, a_1, a_2, ..., a_n$ are real number.

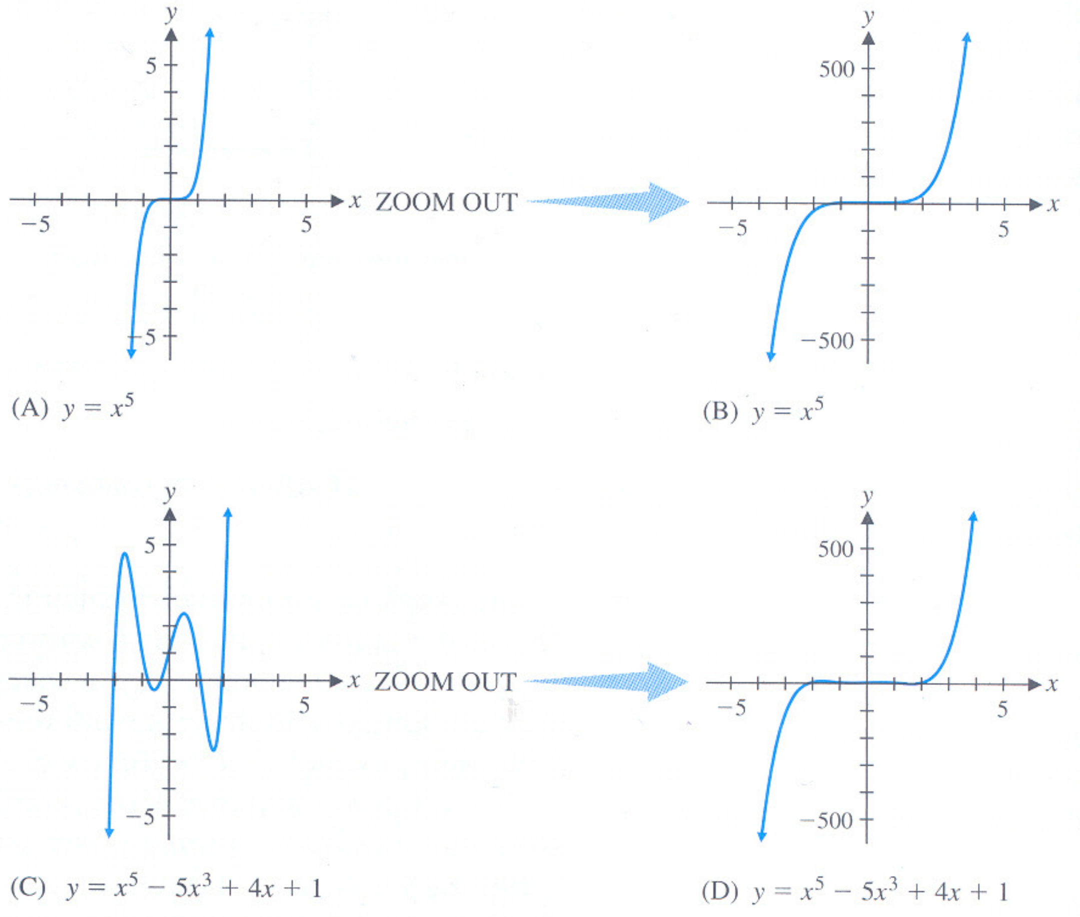
Domain is ________________________.

- The shape of the graph of a polynomial function is connected to the degree of the polynomial.
- Odd-degree polynomial, where the coefficient in front of the highest degree term is positive, starts from negative value, ends with positive value and cross the x axis at least one.
- Even-degree polynomial starts and ends with positive value and may not cross the x axis at all.
- All graphs are continuous and has no shape corner.
- Turning point is the point which separates an increasing portion and decreasing portion.
- The graph of a polynomial function of positive degree n can has at most n-1 turning point and can cross the x axis at most n times.



(A) $f(x) = x - 2$ (B) $g(x) = x^3 - 2x$ (C) $h(x) = x^5 - 5x^3 + 4x + 1$

(D) $F(x) = x^2 - 2x + 2$ (E) $G(x) = 2x^4 - 4x^2 + x - 1$ (F) $H(x) = x^6 - 7x^4 + 14x^2 - x - 5$



2.2.5 Rational Functions

General form:

$$y = f(x) = \frac{N(x)}{D(x)}$$

Where $N(x)$ and $D(x)$ are polynomials.

Domain is ____.

Ex. 17: Define the domain of the following rational functions.

(a) $y = \frac{1}{x^2}$

(b) $y = \frac{x-3}{x^2 - 4x + 4}$

(c) $y = \frac{x^4 - 7}{x}$

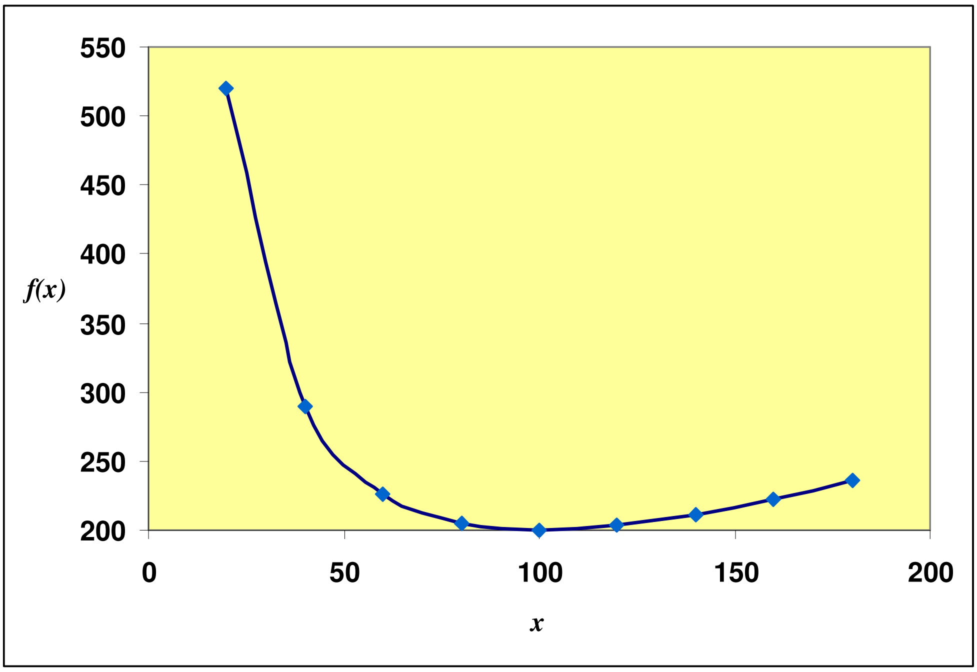
2.3 Functional Models

Practical problems in economics are often too complicated to be precisely described by simple formulas, and one of our basic goals is to develop mathematical methods for dealing with such problem. Toward this end, we shall use a procedure called **mathematical modelling,** which contains four stages: formulation of mathematical model, analysis of the model, interpretation and testing & adjustment.

Ex. 18: The highway department is planning to build a picnic area for motorist along a major highway. It is to be rectangular with an area of 5,000 square yards and is to be fenced off on the three sides not adjacent to the highway. Express the number of yards of fencing required as a function of the length of the unfenced side.

Ans: Let x be the length of the unfenced side (yard), y the length of another side of the rectangular (yard), and F is the required length of fencing (yard).

$$F = x + 2y, \quad xy = 5000 \text{ and } F(x) = x + \frac{10{,}000}{x}$$



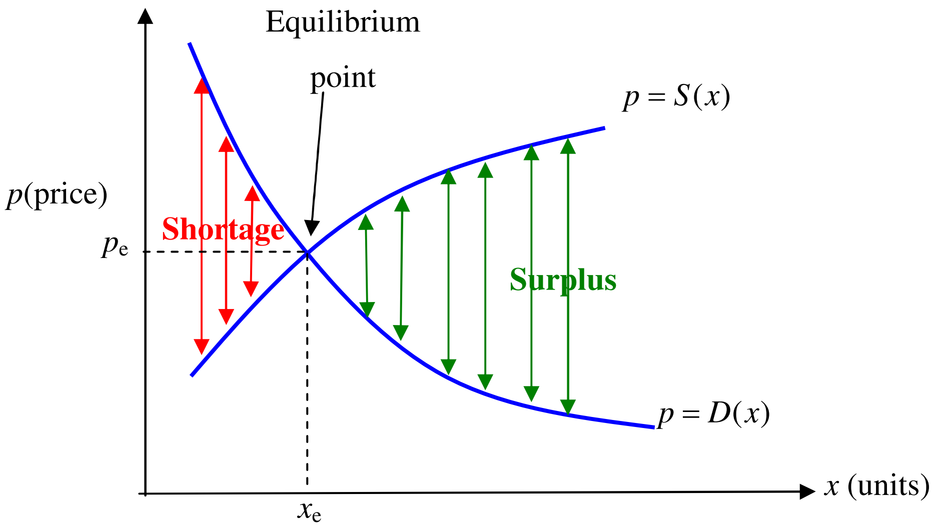
x	$f(x)$
20	520
40	290
60	227
80	205
100	200
120	203
140	211
160	223
180	236

Ex. 19: The rate at which the temperature of an object changes (R) is proportional to the difference between its own temperature (T_o) and the temperature of the surrounding medium (T_e). Express this rate as a function of the temperature of the object.

Ans: $R = k(T_o - T_e)$

2.3.1 Market Equilibrium

Recall the **demand function** $D(x)$ for a commodity relates the number of units x that are produced to the unit price $p = D(x)$ at which all x units are demanded (sold) in the marketplace. Similarly, the **supply function** $S(x)$ gives the corresponding price $p = S(x)$ at which producers are willing to supply x units to the market place. Usually, as the price of a commodity increases, more units of the commodity will be supplied ad fewer will be demanded.



The **law of supply and demand** says that in a competitive market environment, supply tends to equal demand, and when this occurs, the market is said to be in **equilibrium**. Thus, market equilibrium occurs precisely at the production level x_e where $S(x_e) = D(x_e)$. The corresponding unit price p_e is called the **equilibrium price**; that is

$$p_e =$$

When the market is not in equilibrium, it has a **shortage** when demand exceeds supply $[D(x_e) > S(x_e)]$ and a **surplus** when supply exceeds demand $[S(x_e) > D(x_e)]$.

Ex. 20: Market research indicates that manufactures will supply x units of a particular commodity to the marketplace when the price is $p = S(x)$ dollars per unit and that the same number of units will be demanded (bought) by consumers when the price $p = D(x)$ dollars per unit, where the supply and demand functions are given by

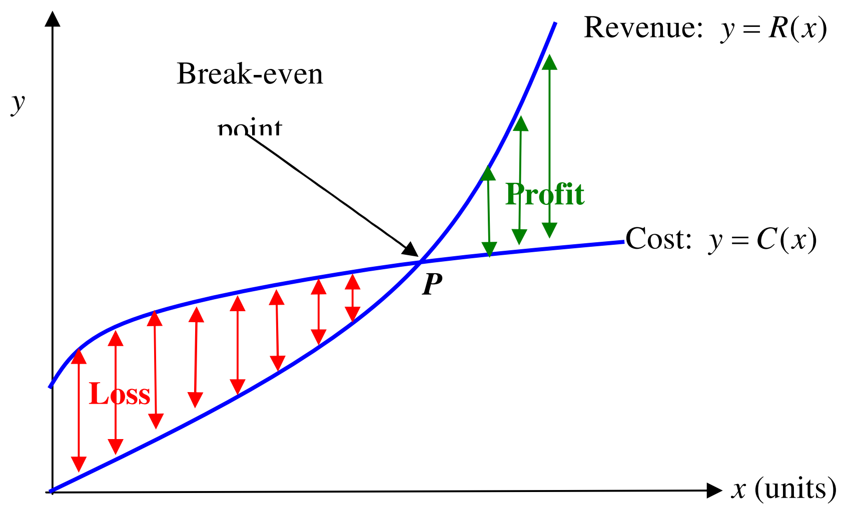
$$S(x) = x^2 + 14 \qquad \text{and} \qquad D(x) = 174 - 6x$$

(a) At what level of production x and unit price p is market equilibrium achieved?

(b) Sketch the supply and demand curve, $p = S(x)$ and $p = D(x)$, on the same graph and interpret.

Ans: (a) 10, $x < 10$ shortage and $x > 10$ surplus, equilibrium at (10,114)

2.3.2 Break-Even Analysis:



In a typical situation, a manufacturer wishes to determine how many units of a certain commodity have to be sold for total revenue to equal to total cost. Suppose x denotes the number of units manufactured and sold, and let $C(x)$ and $R(x)$ be the corresponding total cost and total revenue, respectively. Because of fixed overhead costs, the total cost curve is initially higher than the total revenue curve. Hence, at low levels of production, the manufacturer suffers a loss. At higher levels of production, however, the total revenue curve is the higher one and the manufacturer realises a profit. The point at which the two curves cross is called the **break-even point**.

Ex. 21: A manufacturer can sell a certain product for \$110 per unit. Total cost consists of a fixed overhead of \$7,500 plus production costs of \$60 per unit.

(a) How many unit must the manufacturer sell to break even?

(b) What is the manufacturer's profit or loss if 100 units are sold?

(c) How many unit must be sold for the manufacturer to realise a profit of \$1,250?

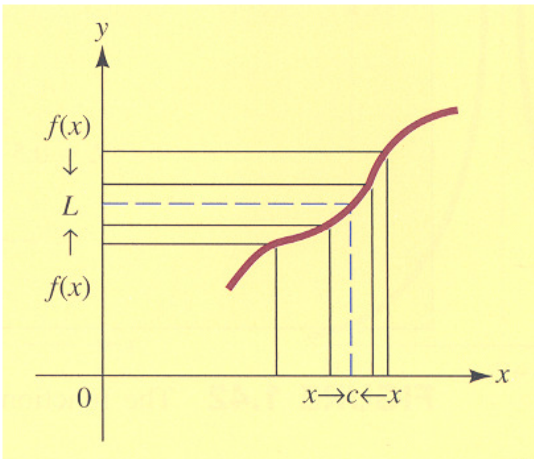
Ans: (a) 150, (b) loss \$2,500, (c) 175

Ex. 22: A certain car rental agency charges \$25 plus 60 cents per mile. A second agency charge \$30 plus 50 cents per mile. Which agency offers the better deal?

Ans: Depending on mileage, 50 miles is the break-even point.

2.4 Limits

Roughly speaking, the limit process involves examining the behaviour of a function $f(x)$ as x approaches a number c that may or may not be in the domain of f.



If $f(x)$ gets closer and closer to a number L as x gets closer and closer to c form both sides, then L is the limit of $f(x)$ as x approaches c.

$$\lim_{x \to c} f(x) = f(c) = L$$

Ex. 23: Use a table to estimate the limit

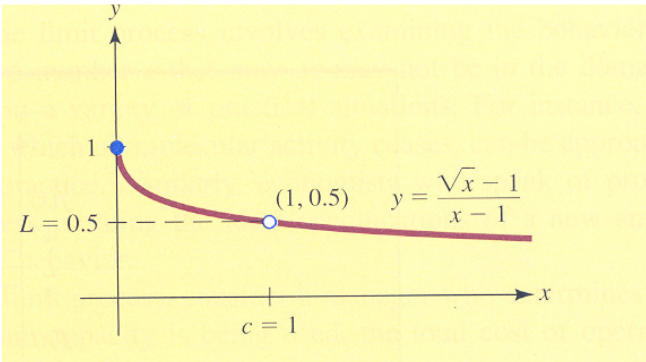
$$\lim_{x\to 1}\left(\frac{\sqrt{x}-1}{x-1}\right)$$

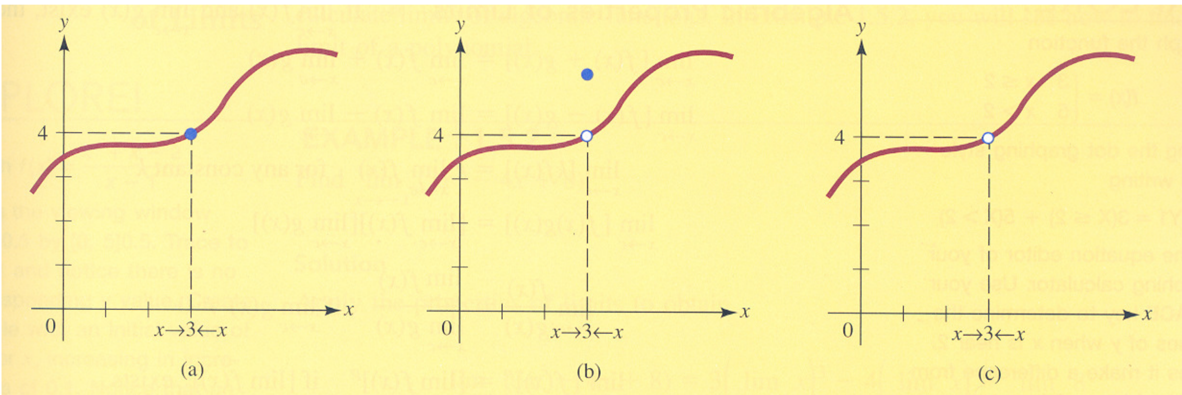
Ans:

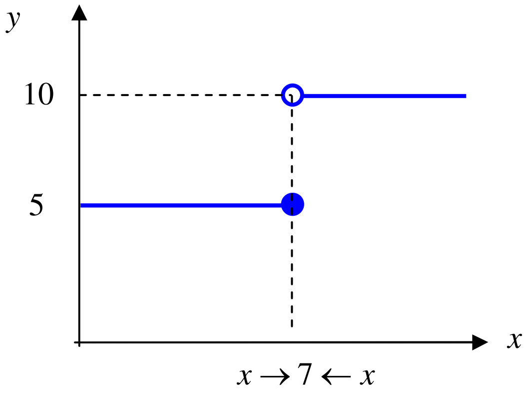
$$x \to 1 \leftarrow x$$

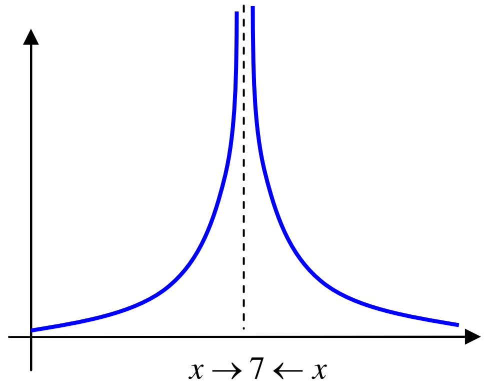
x	0.99	0.999	0.9999	1	1.00001	1.0001	1.001
$f(x)$	0.50126	0.50013	0.50001	**#DIV/0!**	0.499999	0.49999	0.49988

Hence, $$\lim_{x\to 1}\left(\frac{\sqrt{x}-1}{x-1}\right)=0.5$$









Above graphs have no real number limits as x is approaching 7. For the right-hand graph, as x is approaching a number, in this case 7, the y value increases or decrease without bound. Technically, such a limit does not exist, but more information can be given about the behaviour of a function

or

2.4.1 Algebraic Properties of Limits

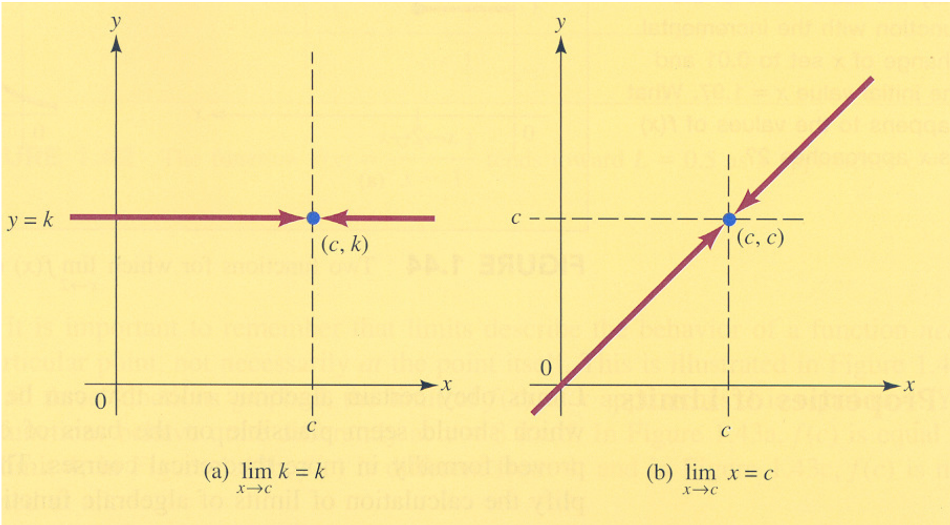
If $\lim_{x \to c} f(x)$ and $\lim_{x \to c} f(x)$ exist, then

- $\lim_{x \to c} \left[f(x) \pm g(x) \right] = \lim_{x \to c} f(x) \pm \lim_{x \to c} g(x)$
- $\lim_{x \to c} \left[kf(x) \right] = k \lim_{x \to c} \left[f(x) \right]$ for any constant k
- $\lim_{x \to c} \left[f(x) g(x) \right] = \left[\lim_{x \to c} f(x) \right] \cdot \left[\lim_{x \to c} g(x) \right]$
- $\lim_{x \to c} \left[\frac{f(x)}{g(x)} \right] =$ if $\lim_{x \to c} g(x) \neq 0$
- $\lim_{x \to c} \left[f(x) \right]^p = \left[\lim_{x \to c} f(x) \right]^p$ if $\left[\lim_{x \to c} f(x) \right]^p$ exists

2.4.2 Limits of Two Linear Functions

For any constant k,

$$\lim_{x \to c} k = k \qquad \text{and} \qquad \lim_{x \to c} x = c$$



Ex. 24: Find the following limits

(a) $\lim_{x \to -1} \left(3x^3 - 4x + 8\right)$ Ans: 9

(b) $\lim_{x \to 1} \left(\dfrac{3x^3 - 8}{x - 2}\right)$ Ans: 5

2.4.3 Limits of Polynomials and Rational Functions

For $p(x)$ and $q(x)$ are polynomials, then

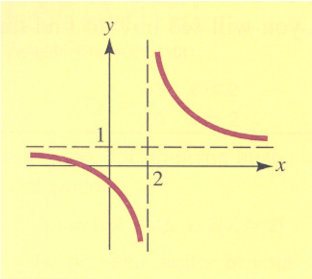
$$\lim_{x \to c} p(x) = p(c)$$

and

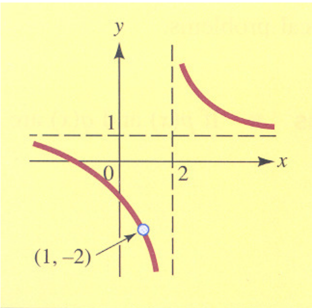
$$\lim_{x \to c} \frac{p(x)}{q(x)} = \frac{p(c)}{q(c)} \qquad \text{if } q(c) \neq 0$$

Ex. 25: Find the following limits

(a) $\lim_{x\to 2}\left(\frac{x+1}{x-2}\right)$ Ans: Not exists.



(b) $\lim_{x\to 1}\left(\frac{x^2-1}{x^2-3x+2}\right)$ Ans: -2



(c) $\lim_{x\to 1}\left(\frac{\sqrt{x}-1}{x-1}\right)$ Ans: 0.5

2.4.4 Limit Involving Infinity

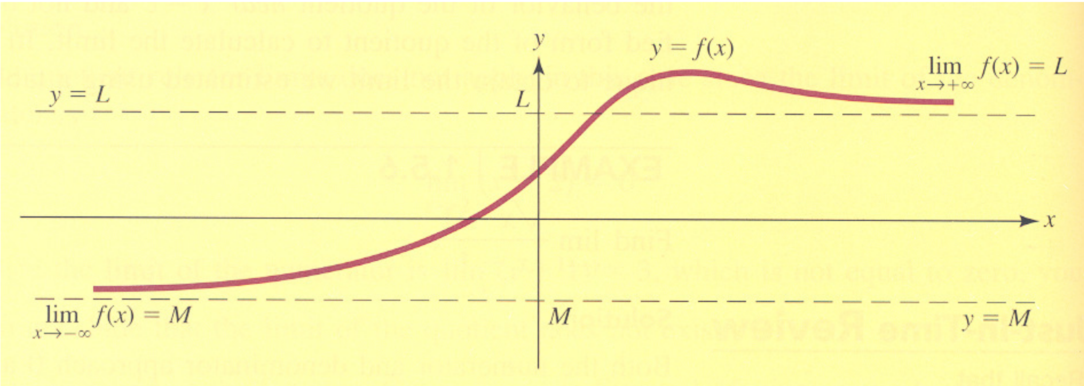
Limit at infinity is the value of the function $f(x)$ approach the number L as x increases without bound,

$$\lim_{x\to+\infty} f(x)=L$$

Similarly, the value of the function $f(x)$ approach the number M as x decreases without bound,

$$\lim_{x\to-\infty} f(x)=M$$

Geometrically, the limit statement $\lim_{x\to+\infty} f(x)=L$ means that as x increases without bound, the graph of $f(x)$ approaches the horizontal line $y=L$, while $\lim_{x\to-\infty} f(x)=M$ means that the graph of $f(x)$ approaches the horizontal line $y = M$ as x decrease without bound. These two lines are called **horizontal asymptotes** of the graph of $f(x)$.



$$\lim_{x\to+\infty} x^k = +\infty \quad \text{and} \quad \lim_{x\to-\infty} x^k = \begin{cases} +\infty & \text{if } k \text{ is even number} \\ -\infty & \text{if } k \text{ is odd number} \end{cases}$$

$$\lim_{x\to+\infty} \frac{A}{x^k} = 0 \quad \text{and} \quad \lim_{x\to-\infty} \frac{A}{x^k} = 0$$

$$\lim_{x\to+\infty} C_0 + C_1 x + ... + C_n x^n = \lim_{x\to+\infty} C_n x^n \quad \text{and}$$

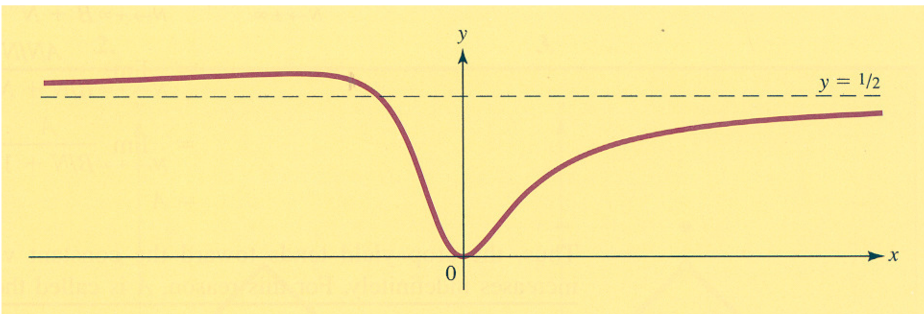
$$\lim_{x\to-\infty} C_0 + C_1 x + ... + C_n x^n = \lim_{x\to-\infty} C_n x^n$$

Ex. 26: Find the following limits

(a) $\lim_{x\to+\infty}\left(\frac{x^2}{1+x+2x^2}\right)$

x	100	1000	10000	100000
$f(x)$	0.49749	0.49975	0.49997	0.499997

$$\lim_{x\to+\infty}\left(\frac{x^2}{1+x+2x^2}\right) = 0.5$$



(b) $\lim_{x\to\infty}\left(\frac{2x^2+3x+1}{3x^2-5x+2}\right)$

Ans: 2/3

Ex. 27: If a crop is planted in soil where the nitrogen level is *N*, then crop yield, *Y*, can be modelled by the *Michaleis-Menten* function:

$$Y(N)=\frac{AN}{B+N} \qquad N\geq 0$$

Where *A* and *B* are positive constants. What would happen to the crop yield as the nitrogen level increased indefinitely?

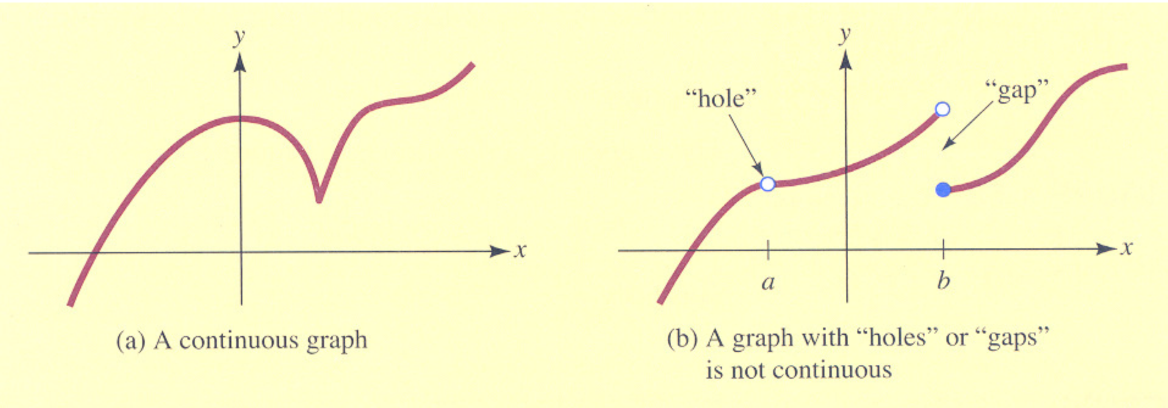
Ans: *A*

Ex. 28: Find the limit $\lim_{x\to\infty}\left(\frac{-x^3+2x+1}{x-3}\right)$

Ans: $-\infty$

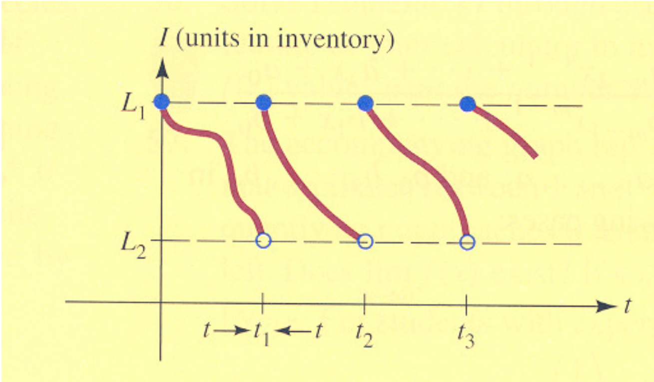
2.5 One-Sided Limits and Continuity

Informally, a *continuous function* is one whose graph can be drawn without the "pen" leaving the paper. A function is not continuous if its graph has a "gap or hole". This gap or hole can be described by a **one-sided limit** of a function; that is, a limit in which the approach is either from the right or left, rather than from both sides as required by the definition of limit (the two-sided limit).



(a) A continuous graph

(b) A graph with "holes" or "gaps" is not continuous



2.5.1 One-Sided Limits:

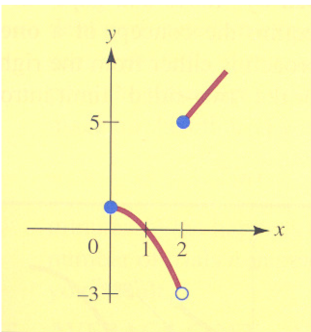
If $f(x)$ approach L_1 as x tends toward c from the left $(x<c)$, we write $\lim_{x\to c^-} f(x) = L_1$.

Likewise, if $f(x)$ approach L_2 as x tends toward c from the right $(x>c)$, we write $\lim_{x\to c^+} f(x) = L_2$.

Ex. 29: For the function

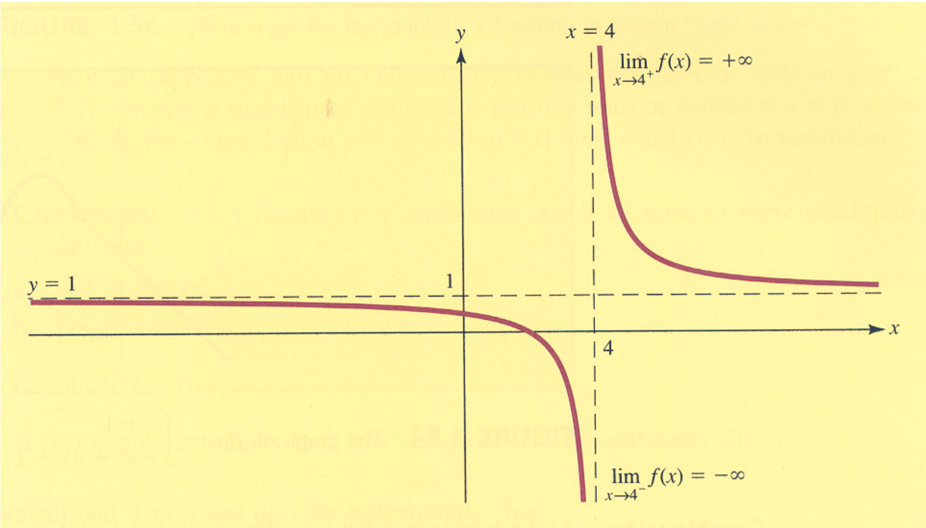
$$f(x) = \begin{cases} 1-x^2 & if \quad 0 \le x < 2 \\ 2x+1 & if \quad x \ge 2 \end{cases},$$

Determine the one-side limits $\lim_{x\to 2^-} f(x)$ and $\lim_{x\to 2^+} f(x)$. Ans; -3, 5



Ex. 30: Find $\lim\left(\frac{x-2}{x-4}\right)$ as x approaches 4 from the left and from the right.

Ans: $-\infty, \infty$



2.5.2 Existence of a Limit:

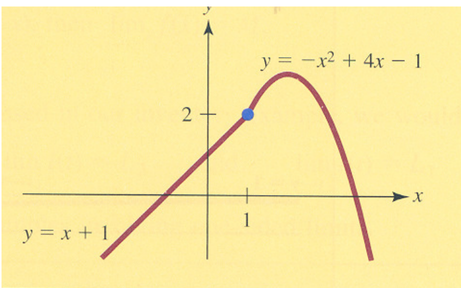
The two-sided limit $\lim_{x\to c} f(x)$ exists if and only if the two one-sided limits $\lim_{x\to c^-} f(x)$ and $\lim_{x\to c^+} f(x)$ **both exist and are equal**, and then

$$\boxed{\lim_{x\to c} f(x) = \lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x)}$$

Ex. 31: Determine whether $\lim_{x\to 1} f(x)$ exists, where

$$f(x) = \begin{cases} x+1 & if \quad x<1 \\ -x^2+4x-1 & if \quad x\geq 1 \end{cases}$$

Ans: exists, $\lim_{x\to 1} f(x) = 2$



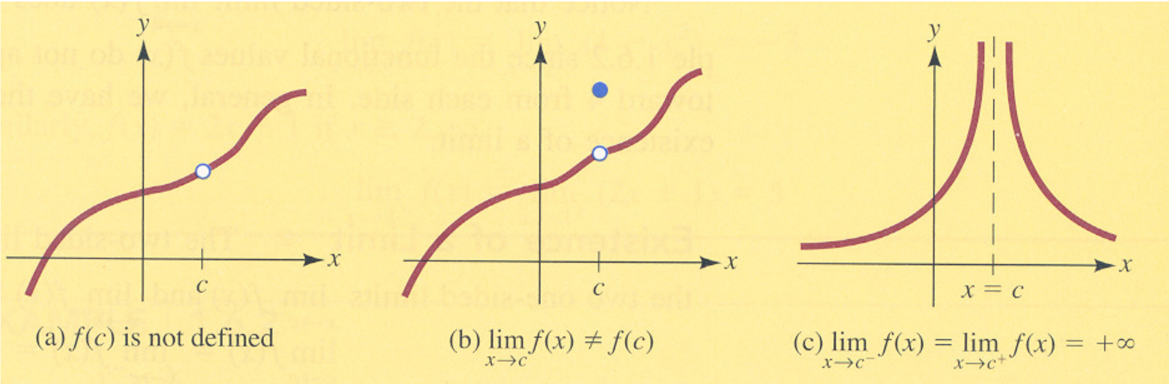
2.5.3 Continuity

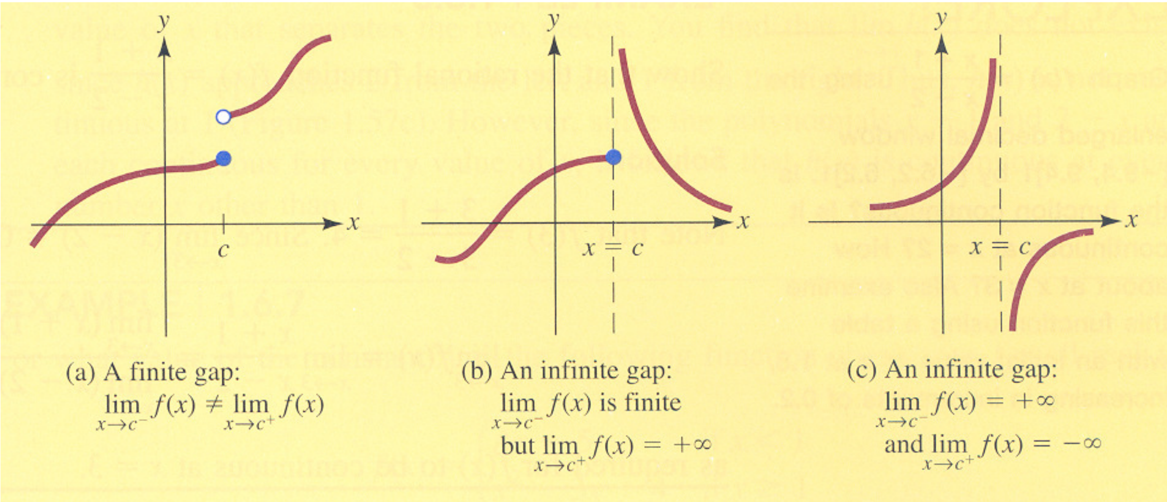
A function *f* is continuous at *c* if all three of these conditions are satisfied:

(1) $f(c)$

(2) $\lim_{x \to c} f(x)$

(3) $\lim_{x \to c} f(x)$





2.5.4 Continuity of Polynomials and Rational Functions

Recall that if $p(x)$ and $q(x)$ are polynomials, then

$$\lim_{x \to c} p(x) = p(c)$$

and

$$\lim_{x \to c} \frac{p(x)}{q(x)} = \qquad \text{if } q(c) \neq 0$$

These limit formulas can be interpreted as saying that **a polynomial or a rational function is continuous wherever it is defined.**

Ex. 32: Show that the polynomial $p(x) = 3x^3 - x + 5$ is continuous at $x = 1$.

Ans: $\lim_{x \to 1} p(x) = 7$

Ex. 33: Show that the rational function $f(x) = \dfrac{x+1}{x-2}$ is continuous at $x = 3$.

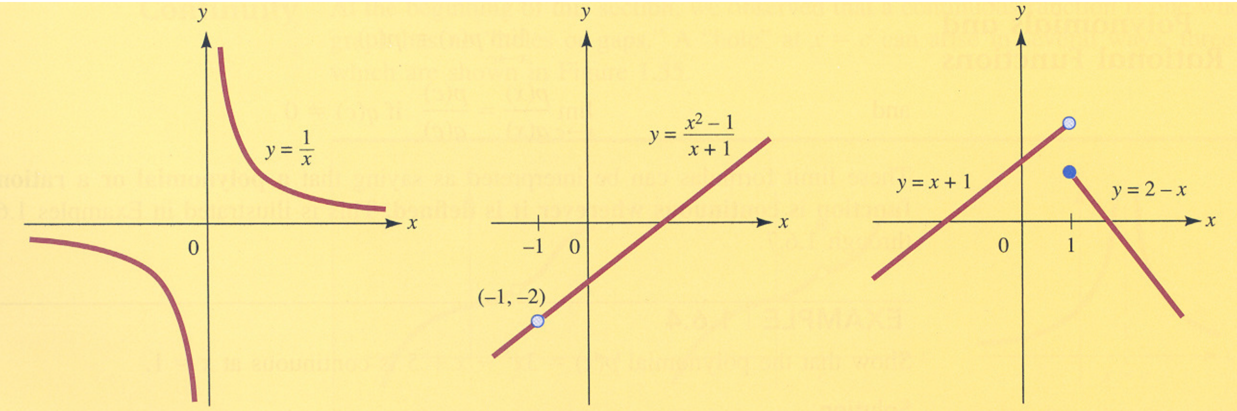
Ans: $\lim_{x \to 3} f(x) = 4$

Ex. 34: Discuss the continuity of the following functions:

(a) $f(x) = \dfrac{1}{x}$ Ans: Continuous for $x \neq 0$

(b) $g(x) = \dfrac{x^2 - 1}{x+1}$ Ans: Continuous for $x \neq -1$

(c) $h(x) = \begin{cases} x+1 & if \quad x < 1 \\ 2-x & if \quad x \geq 1 \end{cases}$ Ans: Continuous for $x \neq 1$



(a) Continuous for $x \neq 0$ (b) Continuous for $x \neq -1$ (c) Continuous for $x \neq 1$

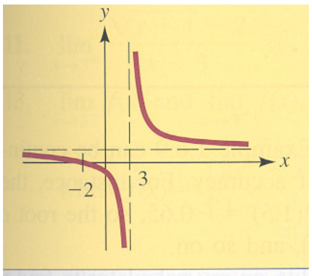
2.5.5 Continuity on an Interval:

A function *f*(*x*) is said to be continuous on an open interval $a < x < b$ if it is continuous at each point $x = c$ in that interval.

Moreover, $f(x)$ is continuous on the closed interval $a \le x \le b$ if it is continuous on the open interval $a < x < b$ and

$$\lim_{x \to a^+} f(x) = \qquad \text{and} \qquad \lim_{x \to b^-} f(x) =$$

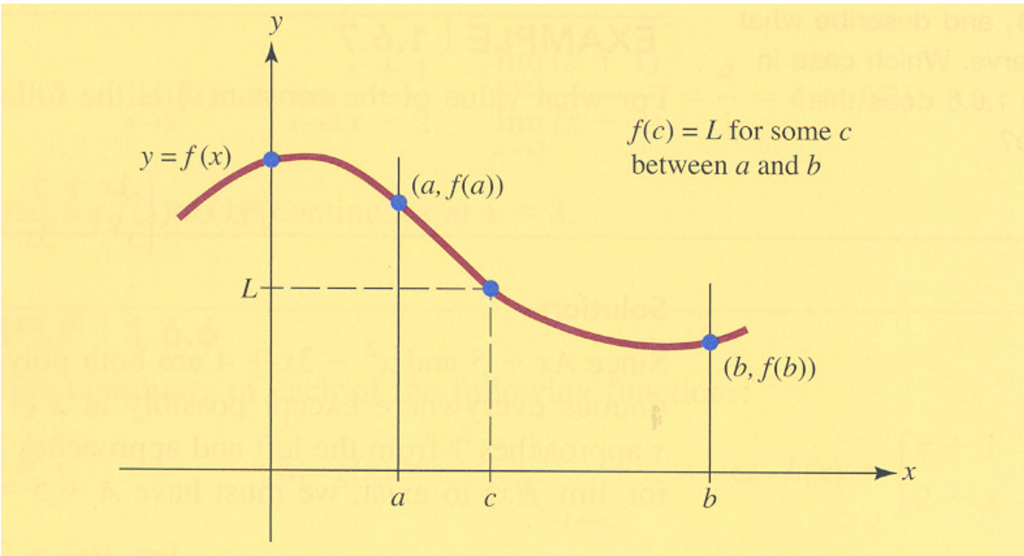
Ex. 35: Discuss the continuity of $f(x) = \dfrac{x+2}{x-3}$ on the open interval $-2 < x < 3$ and on the closed interval $-2 \le x \le 3$.



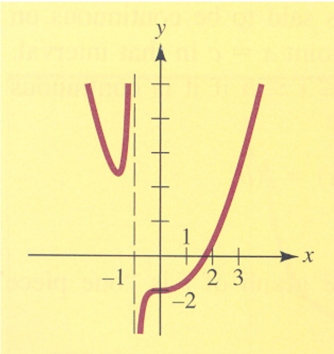
Ans: Continuous for $x \ne 3$. Hence, continuous on the open interval but discontinuous on the closed interval at point $x = 3$.

2.5.6 The Intermediate Value Property:

If $f(x)$ is continuous on the interval $a \le x \le b$ and L is a number between $f(a)$ and $f(b)$, then $f(c)$ = L for some number c between a and b. In other words, *a continuous function attains all values between any two of its values.*



Ex. 36: Show that the equation $x^2 - x - 1 = \dfrac{1}{x+1}$ has a solution for $1 < x < 2$.



2.6 Additional Examples

1. What is the domain of the function $f(t) = \dfrac{\sqrt{t^2 - 1}}{t - 3}$?

   Ans: All t with $t \geq 1$, $t \leq -1$, and $t \neq 3$

2. At a certain factory, the total cost of manufacturing q units during the daily production run is $C(q) = q^2 + 2q + 297$ dollars. On a typical workday, $q(t) = 17t$ units are manufactured during the first t hours of a production run. How many dollars are spent during the first 3 hours of production?

   Ans: 3,000

3. At a certain factory, the total cost of manufacturing units during the daily production run is $C(q) = q^2 + 2q + 254$ dollars. On a typical day, $q(t) = 11t$ units are manufactured during the first hours of a production run. How much is spent during the first 3 hours of production?

   Ans: $1,409

4. Find the composite function $f(2x - 3)$, where $f(x) = \frac{1}{x} - x$.

   Ans: $\frac{-4x^2+12x+-8}{2x-3}$.

5. True or False: The domain of $f(x) = \dfrac{x+1}{x^2-1}$ is all x except 1.

A) True

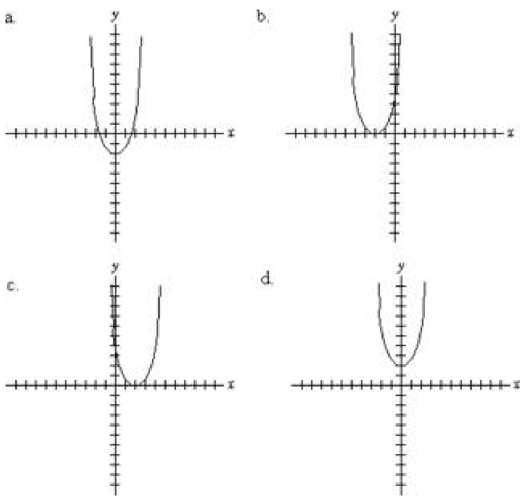
B) False

Ans: B

6. A manufacturer of self-baiting mousetraps is currently selling 1,500 traps a month to retailers at a price of \$1 per trap. She estimates that for each 5 cent increase in price, she will sell 25 fewer traps per month. Her costs consist of a fixed overhead of \$180 a month and 30 cents per trap for labor and materials. Find the dollar profit $P(x)$ as a function of the price x. Estimate the value of x where the maximum occurs from the graph.

Ans: $P(x) = -500x^2 + 2{,}150x - 780,\ x = 2.15$

7. The graph of $f(x) = x^2 + 2$ is



A) Graph a B) Graph b C) Graph c D) Graph d

Ans: D

8. The graphs of $y = x^3$ and $y = x$ intersect at

A) (1, 1) and (–1, –1)

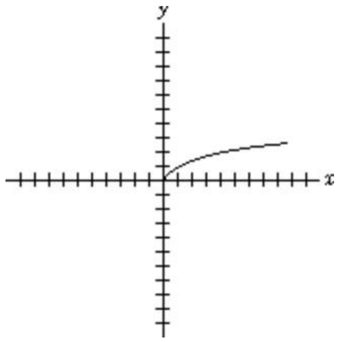
B) (1, 1)

C) (1, 1), (–1, –1), and (0, 0)

D) (1, 1) and (0, 0)

Ans: C

9. True or False: The graph of $f(x) = \frac{1}{\sqrt{x}}$ is shown below.



A) True

B) False

Ans: B

10. An apartment complex in a small college town has 120 units total. If they charge $1200 per month, they can rent every apartment, but each $50 increase in rent will result in 3 vacant apartments. Find the price they should charge to maximize their revenue. (Hint: Write a function describing revenue in terms of monthly rent, then find the vertex).

Ans: $1,600.00

11. Find the slope and y-intercept of the line $3x – 5y = 9$.

Ans: 3/5, 9/5

12. Find the slope and y-intercept (if they exist) of the line $5y = 8x$.

A) slope 8, y-intercept 0

B) slope 8, y-intercept 5

C) slope $\frac{8}{5}$, y-intercept 0

D) slope $-\frac{8}{5}$, y-intercept 0

Ans: C

13. A concert promoter estimates that if he sells tickets at a price of $15 each, he will sell 200 tickets. For each dollar that he decreases the price he will sell 20 more tickets. Express the total revenue R from ticket sales as a function of the ticket price p. Graph the result and estimate the optimum ticket price.

Ans: $R = p(200 + 20(15 – p)) = 500p – 20\,p^2$, p=$12.50

14. A company makes a certain product for \$4 each and sells it for \$8. If the company has overhead expenses of \$10,000 per year, how many of its products must be made and sold to break even?

    A) 10,000 B) 20,000 C) 40,000 D) 2,500

    Ans: D

15. A manufacturer's total cost consists of a fixed overhead of \$100 plus production costs of \$10 per unit. Express the total cost in dollars as a function of the number of units produced.

    Ans: $C(x) = 10x + 100$.

16. Find the limit: $\lim_{x\to 1} \frac{2x-1}{x+3}$

    Ans: 1/4

17. Find the limit: $\lim_{x\to 2} \frac{x-2}{x^2-4}$

    Ans: 1/4

18. Find the limit: $\lim_{x\to 4} \frac{\sqrt{x}-2}{x-4}$

    A) Does not exist B) 4 C) $-\frac{1}{4}$ D) $\frac{1}{4}$

    Ans: D

19. Find the limit as $x \to 11$ of $\frac{\sqrt{5x-30}-6}{x-11}$

    A) 0 B) 5 C) $\frac{1}{2}$ D) The limit does not exist.

    Ans: D

20. Find the limit: $\lim_{x\to 3} \frac{x-3}{x^2-9}$

    Ans: 0.17

21. True or False: $\lim_{x\to 1}\frac{\sqrt{x}-1}{x^2-1}$ does not exist.

A) True

B) False

Ans: B

22. Find the limit: $\lim_{x\to 1^+}\frac{\sqrt{x-1}}{x-1}$

Ans: Infinity

23. Find all values of *x* for which the function is not continuous. $f(x)=\frac{x^2+3x+2}{x^2+4x+3}$

Ans: –1,–3

24. For which value of *x* is the following function not continuous? $f(x)=\begin{cases} x-2 & \text{if } x<2 \\ 1 & \text{if } x=2 \\ 2-x & \text{if } x>2 \end{cases}$

A) 1 B) 2 C) 0 D) –2

Ans: B

25. Find all the values of *x* for which the function is not continuous.

$$f(x)=\begin{cases} x^2+4 & \text{if } x\le 5 \\ 4 & \text{if } x>5 \end{cases}$$

Ans: 5

26. Find the limit as $x\to 16^-$ of $\frac{\sqrt{x}-4}{x-16}$.

Ans: $\frac{1}{8}$

27. True or False: The following function is continuous at $x = 1$:

$$f(x) = \begin{cases} x^2 - x + 1 & \text{if } x < 1 \\ 1 & \text{if } x = 1 \\ \frac{x^2-1}{x-1} & \text{if } x > 1 \end{cases}$$

A) True

B) False

Ans: B

28. True or False: $\lim_{x \to 3^-} f(x) = 3$, where $f(x) = \begin{cases} x & \text{if } x < 3 \\ x+1 & \text{if } x \geq 3 \end{cases}$

A) True

B) False

Ans: A

29. Find all values of c that make the function $f(x)$ continuous for all x.

$$f(x) = \begin{cases} cx^2 - 5 & x \leq 2 \\ 2x + 3 & x > 2 \end{cases}$$

Ans: $c = 3$