Solution: Quiz 5

1. Define $H : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ as follows:

$$H(a,b) = (3a^2 - 1, 4b+1) \text{ for all } (a,b) \in \mathbb{Z} \times \mathbb{Z},$$

where $\mathbb{Z}$ is the set of all integers.

   (a) Is H one-to-one? Prove or give a counterexample.

   (b) Is H onto? Prove or give a counterexample.

   (c) Is H bijective? If so, find H^{-1}, the inverse function of H.

Solution:

(a) Is H one-to-one? Prove or give a counterexample.
Answer: No, H is not one-to-one. A counterexample is when we choose two different elements $(-1,0)$ and $(1,0)$ from the domain $\mathbb{Z} \times \mathbb{Z}$, we have $H(-1,0) = H(1,0)$, i.e.,

$$H(-1,0) = (3(-1)^2 - 1, 4 \cdot 0 + 1) = (2,1)$$

and

$$H(1,0) = (3(1)^2 - 1, 4 \cdot 0 + 1) = (2,1).$$

(b) Is H onto? Prove or give a counterexample.
Answer: No, H is not onto. Notice that if we pick (u,v) from the co-domain $\mathbb{Z} \times \mathbb{Z}$ and suppose that there is (a,b) in the domain such that $H(a,b) = (u,v)$, then

$$(u,v) = H(a,b) = (3a^2 - 1, 4b+1)$$

or we must have

$$u = 3a^2 - 1 \quad \Rightarrow \quad a = \pm\sqrt{\frac{u+1}{3}}$$

and

$$v = 4b + 1 \quad \Rightarrow \quad b = \frac{v-1}{4}.$$

So, the following is a counterexample for this. If we choose $(u,v) = (0,0)$ from the co-domain $\mathbb{Z} \times \mathbb{Z}$, then, in order to have $H(a,b) = (0,0)$, we must set $a = \sqrt{\frac{0+1}{3}} = \sqrt{\frac{1}{3}}$ or $a = -\sqrt{\frac{0+1}{3}} = -\sqrt{\frac{1}{3}}$, and set $b = \frac{0-1}{4} = \frac{-1}{4}$. However, $a = \pm\sqrt{\frac{1}{3}}, b = -\frac{1}{4}$ are not in $\mathbb{Z}$ and therefore, we **cannot** find any element (a,b) from the domain $\mathbb{Z} \times \mathbb{Z}$ such that $H(a,b) = (0,0)$. That is, H is not onto.

(c) Is H bijective? If so, find H^{-1}, the inverse function of H.
Answer: No, H is not bijective because it is neither one-to-one nor onto. So we cannot find its inverse function.