EE422 (1/2013)
Mathematical Economics 2
Dynamic Programming

Read Martin Boileau's paper

topics

- 1. Introduction and value function
- 2. Bellman's equation and Principle of Optimality
- 3. Solving DP: Method of value function iteration : A cake-eating problem
- 4. Solving DP: Guess and verify method: Brock and Mirman, Optimal growth problem
- 5. Benveniste-Scheinkman Formula

Readings

- Ian King (2002) A Simple Introduction to Dynamic Programming in Macroeconomic Models.
- Martin Boileau (?) A Child's Guide to Dynamic Programming.
- Williamson. Notes on Macroeconomic Theory.
- Sargent, Thomas (1987). Dynamic Macroeconomic Theory.
- Ljungqvist, Lars and Thomas Sargent (2004). Recursive Macroeconomic Theory, 2nd edition.
- Stokey, Nancy and Robert Lucas (1989) Recursive Methods in Economic Dynamics.

1. Introduction: Dynamic Programming

- Dynamic problem is when the objective function is defined over a sequence of periods and if the agent's choices in one period affect the values of the state variables in the subsequent periods.
- If not, it is just a sequence of static problems.
- Method of optimal control (Maximum principle) is convenient for continuous-time variable.
- Method of Dynamic Programming (Bellman 1957) is for discrete-time variable
  - useful for dynamic optimization with uncertainty
  - numerical application possible

1. Introduction: Dynamic Programming

Intuition:

- Convert a T-period problem into a two-period problem with modifications of the objective function, i.e. Using the value function obtained from solving a shorter horizon problem. Thus, the whole sequences of decisions can be broken into 2 parts: immediate decision, and remaining decisions summarized by a value function.

- DP or Bellman equation allows us to solve larger problem by solving the smaller problem sequentially backward.

1. Introduction: Dynamic Programming

- Consider a finite horizon problem with discrete time 0,…,T

$$\max \quad F(u_0,..,u_T;x_0,..,x_T)$$

$$subject\,to$$

$$\begin{aligned}
&x_{t+1} = g(x_t,u_t), \quad t = 0,\dots,T \\
&x_0 \quad given \\
&x_T \geq 0 \\
&\{u\} \in U.
\end{aligned}$$

1. Introduction: Dynamic Programming

- Our general problem is to maximize $F(u_0,u_1,..,u_T\,;x_0,x_1,\ ..,x_T)$ by selection values of two sequences
- Let $\{u_t\}$ or $(u_0,u_1,..,u_T)$ a sequence of controls (we must choose its values). Ex. Consumption levels.
- Let $\{x_t\}$or $(x_0,x_1,\ ..,x_T)$, a sequence of states that is observed at discrete time $t=0,\ 1,\ \ldots,\ T$. Its values are affected by controls and states. Ex. Assets or capital stocks.
- State variable is a stock variable, telling you the status of the system (values of capital stock of the economy, your wealth status at any particular time).
- Control variable is a flow variable: its value is determined within the period (what you spend or save today, what firms invest within a year).

1. Introduction: Dynamic Programming

- States of the system evolves through time under the influence of a sequence of controls, u_t by the difference equation called "transition equation"

$$x_{t+1} = g(t, x_t, u_t), \quad t = 0,..,T. \qquad (1)$$

- Explicitly, we also assume that g is time-separable. That is, x_{t+1} depends on time t, state x_t and u_t. (choose u_t after knowing x_t). This implies the control u_t influence state x_{t+1+s} and returns for $s \geq t$, but not earlier.

$$Ex.\, x_1 = g(x_0, u_0);\; x_2 = g(x_1, u_1);\ldots\; x_0 \; given.$$

- Notice that interaction between control and state appear within the same period.
- Ex. Your wealth at the end of the day is determined by your today spending and your morning wealth. And what you have and what you spend today will also affect your tomorrow wealth but not your past.

1. Introduction: Dynamic Programming

- Initial condition: x_0 is given.
- Terminal condition or end-constraint : $x_T \geq 0$ or sometimes can be free. Ex. You can die with some wealth but not with debt.
- This problem can be solved by Lagrange method.
- Solving the system of equations (from first-order conditions) give you values of controls, states, and Lagrange multipliers.
- For large finite time T, this becomes messy.
- We try to simplify this general problem by making some assumptions.

1. Introduction: Dynamic Programming

- First, assume that F is time-separable and additive: we rewrite as $F_0(u_0,x_0)+\ldots+F_T(u_T,x_T)$. This allows us to break the problem into smaller ones.
- Our objective fn. becomes

$$\sum_{0}^{T} F_t(x_t, u_t)$$

- So, $F_t(x_t,u_t)$ shows the value obtained at each t.
- The value associated with a given path is from the summation of F.

1. Introduction: Dynamic Programming

$Suppose\ we\ choose\ values\ for\ u_0, u_1, \ldots, u_{T-1}.$

$For\ a\ given\ (u_0, \ldots, u_{T-1}),\ we\ can\ find\ from\ (1)\ that$

$x_1 = g(0, x_0, u_0).\ Once\ known,\ we\ can\ find$

$x_2 = g(1, x_1, u_1),$

$x_3 = g(2, x_2, u_2), \ldots.$

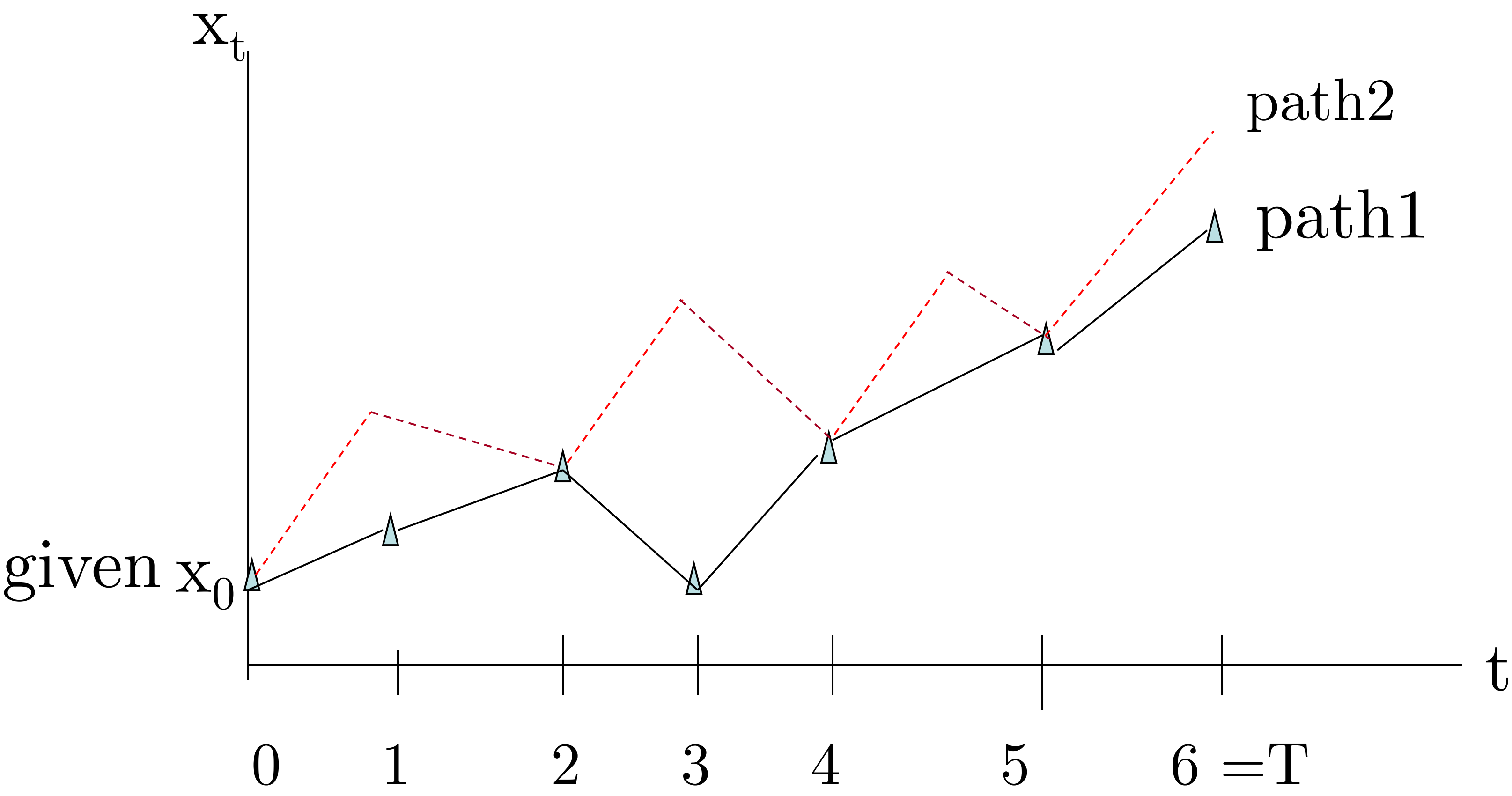
$We\ solve\ them\ recursively\ or\ successively\ for\ a\ sequence$

$(x_1, \ldots, x_T)\ \ in\ terms\ of\ the\ initial\ state,\ x_0\ ,\ and$

$the\ time\ path\ of\ control\ (u_0, \ldots, u_{T-1}).$

$A\ different\ choice\ of\ \ (u_0, \ldots, u_{T-1})\ gives\ another\ path.$

1.Introduciton: Dynamic Programming



1. Introduction: Dynamic Programming

- Our time-separable, recursive problem becomes

$$\begin{aligned} \max \quad & \sum_{t=0}^{T} F_t(x_t, u_t) && (P) \\ \text{s.t.} \quad & x_{t+1} = g_t(x_t, u_t), && ....(1) \\ & x_0 \; given, \;\; u_t \in U. \end{aligned}$$

- Our goal is to find an <u>optimal</u> pair $(\{x_t^*\}, \{u_t^*\})$.
- Optimal control u tells you about <u>decision rule</u> or "<u>policy</u>" which must appear as a function of x *at the beginning period.*
- Ex. Spend only a fraction of your money on any days.

1. Introduction: Dynamic Programming

$Example: At\, each\, t, you\, decide\, how\, much\, to\, consume\, c_t$
$out\, of\, your\, wealth\, a_t.$

$$\max_{\{c\}_0^2,\{a\}_1^2} \sum_{t=0}^{2} \beta^t u(c_t) = \ln(c_0) + \beta \ln(c_1) + \beta^2 \ln(c_2)$$

$$st. \quad a_{t+1} = (1+r)a_t - c_t \;\; for\, t = 0,1,2$$

$$a_0 \; given \;\; and \;\; a_3 = 0.$$

$Find\, optimal\, path\, of\, consumption\, (c_0, c_1, c_2)\, and\, (a_1, a_2).$

-We can solve this using Lagrange method.

(I leave this for your homework).

1. Introduction: value function

- Back to solve (P), we *first* look for intuition of DP.

- We will split the problem into sequences of optimizations.

– Think if we work <u>backward</u> by solving one-period problem and get its value function* W_1, then use it to solve two-period problem, until we get $(T+1)-$ period value function.

– This procedure gives us W_{T+1} as a function of initial state x_o.

– Along the way, we found optiaml policy rules $\{u_t\}$ which solves our original problem.

- If DP works, it suggests the conditions from these recursive backward *patterns* yields the same marginal conditions as in Lagrange method.

1. Introduction: value function

$First, working\, backward\, for\, one - period\;\; problem, t = T:$

$$W_1(x_T) = \max[F_T(x_T, u_T) + W_0(x_{T+1})].$$

$For\, 2 - period\, problem, t = T - 1:$

$$W_2(x_{T-1}) = \max[F_{T-1}(x_{T-1}, u_{T-1}) + W_1(x_T)].$$

$....$

$For\, T + 1 period\, problem, t = 0:$

$$W_{T+1}(x_0) = \max[F_0(x_0, u_0) + W_T(x_1)].$$

$Thus, we\, can\, write\, Bellman\, equation\, of\, this\, sequencial\, problems\;\; as$

$$\boxed{W_{T-t+1}(x_t) = \max[F_t(x_t, u_t) + W_{T-t}(x_{t+1})],\;\; t = 0,..,T.}$$

note: subscript of W=time-remaining period.

At time 0, there are $T + 1$ periods left. Solution for $(T + 1)$ period problem comes from maximizing the immediate F and the maximized *optimal* value of subsequent T problem.

1. Introduction: value function

-Solving this successively will give us

$$u_t = h_t(x_t), \quad t = 0,..,T; and$$
$$x_{t+1} = f_t(x_t), \; t = 1,...,T.$$

—Controls that depend on the state in this way, a function of the state variable at the start of that period, are called "feedback controls", or "policies".

—also called a close-loop solution in optimal control literature.

So, we will look for optimal policy function h_{t}.

-These optimal policies are self-enforcing or time-consistent, meaning that as times come, there is no incentive to deviate from this original plan.

-This is a result of backward induction.

1. Introduction: value function

Principle of Optimality:

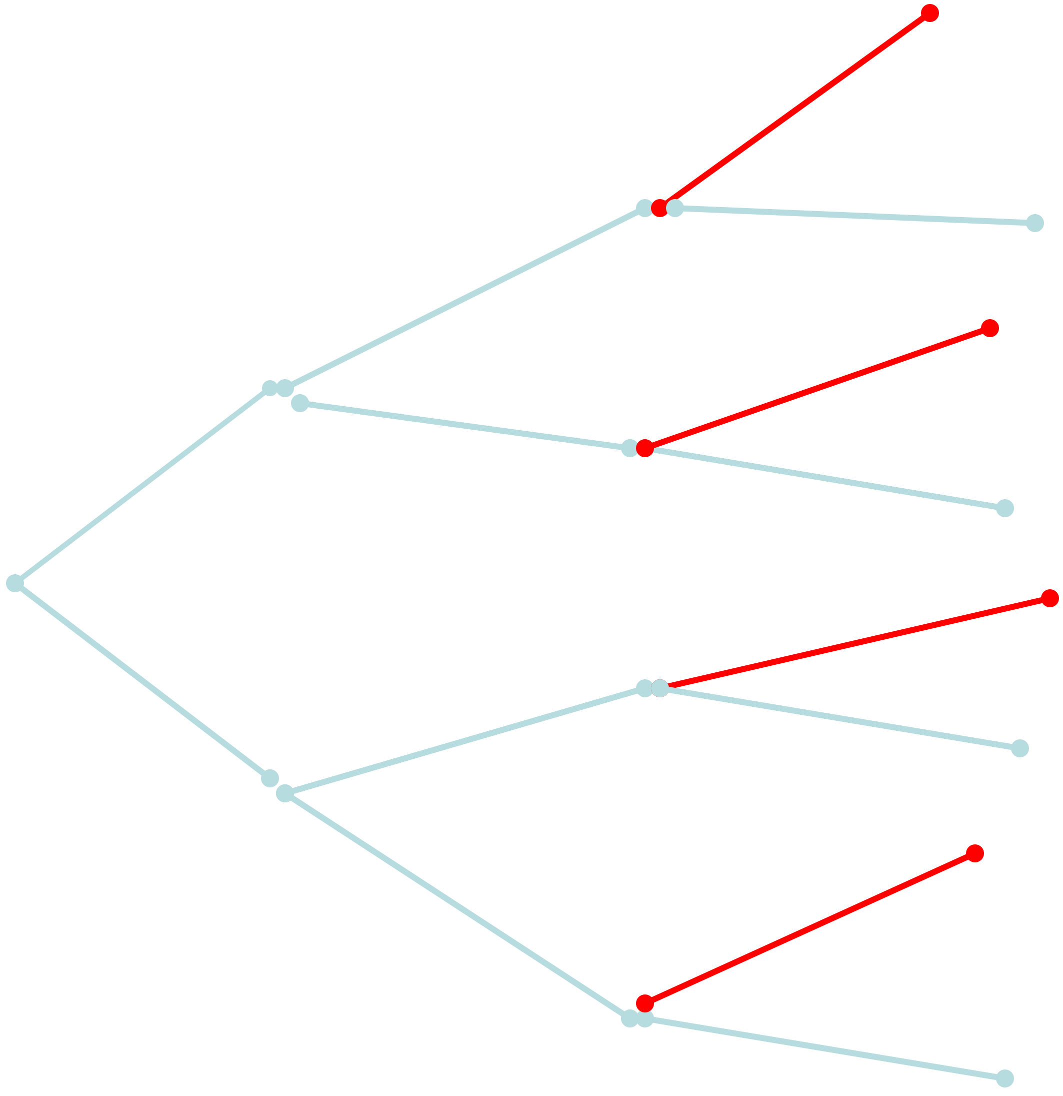
"If the rule for $1-$ period problem is optimal

for given x, then it must be optimal for x of larger problem."

or

"An optimal policy has the property that, whatever the inital action, the remaining choices constitutes an optimal policy with respect to the subproblem starting at the state that results from the initial actions."

Thus, in Bellman eq. we need only to choose immediate control optimally.



2. Bellman's Equation of Dynamic Programming

It is difficult to solve Bellman equation because we look for the function W(.) that satisfy the equation. (functional equation problem)

To solve the Bellman equation of finite DP, we start backward (since we know when or the last period) using this following procedure.

1. Write the Bellman $W_1(x)$ of one-period problem, then solve for $u_T^* = h_T(x_T)$ using standard static opt. method.
2. Write the Bellman $W_2(x)$ *of* $2-$ period problem using information from 1, then solve for $u_{T-1}^* = h_{T-1}(x_{T-1})$.
3. Work backward to get all $W_1(x), \ldots, W_{T+1}(x)$ *and* $u_T^*, \ldots, u_0^*$.

2. Bellman Equation of Dynamic Programming

$$Example : consumption\ over\ time.$$

$$\max_{\{c\}_0^2, \{a\}_1^2} \sum_{t=0}^{2} \beta^t u(c_t) = \ln(c_0) + \beta \ln(c_1) + \beta^2 \ln(c_2)$$

$$st. \quad a_{t+1} = (1+r)a_t - c_t \ \ for\ t = 0,1,2$$

$$a_0\ given \ \ and \ \ a_3 = 0.$$

$$Find\ optimal\ path\ of\ consumption\ (c_0, c_1, c_2)\ and\ (a_1, a_2).$$

2. Bellman Equation of Dynamic Programming

$First, we\ simplify\ the\ problem.$

$Since\ a_3 = 0, thus\ c_2 = (1+r)a_2. Put\ this\ in\ the\ obj\ fn.$

$gives$

$$\max_{\{c\}_0^1, \{a\}_1^2} \ln(c_0) + \beta \ln(c_1) + \beta^2 \ln((1+r)a_2)$$

$$st. \quad a_{t+1} = (1+r)a_t - c_t \quad for\ t = 0,1$$

$$a_0\ given.$$

$Find\ optimal\ path\ of\ consumption\ (c_0, c_1)\ and\ (a_1, a_2).$

$W_2(a_0)$ $W_1(a_1)$

a_0 a_1

t

0 1

As a starting point, $W_0(a_2) = \beta^2 \ln((1+r)a_2)$.

STEP 1. Write Bellman for one-period problem

$$For\; t = 1,\; W_1(a_1) = \max[\beta \ln(c_1) + W_0(a_2)]$$

$$st.\;\; a_2 = (1+r)a_1 - c_1.$$

Next, substitute transition function into the Bellman

$$Thus, \qquad W_1(a_1) = \max[\beta \ln(c_1) + W_0((1+r)a_1 - c_1)]. \quad (*)$$

Next, find first-order-condition, FOC, w.r.t c_1:

$$\frac{\beta}{c_1} - W_0' = 0$$

$$[but\; W_0' = \tfrac{\partial W_0(a_2)}{\partial a_2} = \beta^2 / a_2] \quad \frac{\beta}{c_1} - \frac{\beta^2}{a_2} = 0$$

$$with \qquad a_2 = (1+r)a_1 - c_1$$

$$yields \;\; c_1 = h_1(a_1) = (1+r)(\tfrac{1}{1+\beta})a_1$$

$$and \qquad a_2 = f_1(a_1) = (1+r)(\tfrac{\beta}{1+\beta})a_1$$

sub c_1 *and* a_2 into the Bellman gives

$$W_1(a_1) = \beta \ln\left(\frac{1+r}{1+\beta}\right) a_1 + \beta^2 \ln\left(\frac{\beta}{1+\beta}\right) (1+r)^2 a_1.$$

STEP 2. Write Bellman for 2-period problem

$$For\ t = 0^*,\ W_2(a_0) = \max\left[\ln(c_0) + W_1(a_1)\right]$$

$$st.\ \ a_1 = (1+r)a_0 - c_0.$$

$$= \max\left[\ln(c_0) + W_1((1+r)a_0 - c_0)\right]$$

$$FOC\ wrt\ c_0\ is \qquad \frac{1}{c_0} - W_1' = 0$$

$$\left[but \quad W_1' = \frac{\partial W_1(a_1)}{\partial a_1} = \frac{\beta}{a_1} + \frac{\beta^2}{a_1} = \beta(1+\beta)\frac{1}{a_1}\right.$$

$$or\, W_1' = (1+r)W_0' \qquad \because \text{using } (^*)$$

$$= (1+r)\beta^2 / a_2$$

$$\left. = \beta(1+\beta)\frac{1}{a_1} \quad \because \text{using policy rule } a_2 = f_1(a_1)\right]$$

$$\frac{1}{c_0} - \beta(1+\beta)\frac{1}{a_1} = 0$$

$$with \qquad a_1 = (1+r)a_0 - c_0$$

$$yields \;\; c_0 = h_0(a_0) = (1+r)\;\tfrac{1}{1+\beta(1+\beta)}\;a_0$$

$$and \qquad a_1 = f_0(a_0) = (1+r)\;\tfrac{\beta(1+\beta)}{1+\beta(1+\beta)}\;a_{0.}$$

$$We\,can\,find\,W_2(a_0) = \ln(c_0^*) + W_1(a_1^*) = \ldots$$

$$At\,each\,t, you\,decide\,how\,much\,to\,consume\,c_t$$

$$out\,of\,your\,wealth\,a_t.$$

$$Homework.(value\,function\,iteration)$$

$$\max \sum_{t=0}^{3}(1+x_t-u_t^2)$$

$$st. \quad x_{t+1}=x_t+u_t, t=0,1,2.$$

$$x_0=0,\ u_t\in\mathbb{R}.$$

$$answer.$$

$$W_4(x_0=0)=7.5$$

$$u_3^*=0, u_2^*=0.5, u_1^*=1, u_0^*=1.5$$

2. Bellman Equations for Infinite Horizon DP

-The fact that time is finite, and functions f and g varies over time, we will obtain a different policy function $u_t = h_t(x_t)$.

-For practical applications in economics, we look for the DP structure which yield the time-invarinat policy function.

-DMT p. 20 shows that if we assume

$$F_t(x_t, u_t) = \beta^t f(x_t, u_t), \quad 0 < \beta < 1$$

$$g_t(x_t, u_t) = g(x_t, u_t)$$

-The limiting value function that yields what we want is the value function for the infinite horizon problem.

2. Bellman Equations for Infinite Horizon DP

next, we look at infinite horizon problem, avoiding to say what happens when the finite horizon is reached.

Let $\beta \in (0,1)$ the discount factor, for all t.

Define $F_t(x_t, \ u_t) = \beta^t f(x_t, u_t)$.

Our infinite horizon problem is

$$\begin{aligned} &\max \ \sum_{t=0}^{\infty} \beta^t f(x_t, u_t), \\ &s.t. \quad x_{t+1} = g(x_t, u_t), \ \ t = 0, 1, 2, \ldots, \\ &\qquad x_0 \ \text{given}, \qquad u_t \in U \subseteq R \qquad \ldots.(P1). \end{aligned}$$

2. Bellman Equation for Infinite Horizon DP

–Intuitively, future at any time s looks exactly the same as time 0 for infinite horizon. This is recursive (infinitely repeating) nature: each decision leads to another problem that look exactly like the original one. (stationary of the problem)

- thus calendar date t or s by itself has no effect. This leads to Bellman equation being independent from time t as such. Thus, the value function is common to all periods, although it will be evaluated at different state x_i

–Thus, your *optimal* decision rule must be time-invariant.

–Along with $0 < \beta < 1$, some assumptions to make sure about boundedness of the objective functions : $0 \leq f(x,u) \leq k$.

–The existence of the maximum is attained for f, g are continuous and U is compact.

2. Bellman Equation for Infinite Horizon DP

– from the Bellman equation

$$W_j(x_t) = \max[\beta^t f(x_t, u_t) + W_{j-1}(x_{t+1})]$$

– We can write this in current value by first define the current value function

$$V_j(x_t) = \beta^{-t} W_j(x_t)$$

– Substitute $W_j(x_t)$ into above gives

$$\begin{aligned} V_j(x_t) &= \max[\beta^{-t}\{\beta^t f(x_t, u_t) + W_{j-1}(x_{t+1})\}] \\ &= \max[f(x_t, u_t) + \beta\beta^{-(t+1)} W_{j-1}(x_{t+1})] \\ &= \max[f(x_t, u_t) + \beta V_{j-1}(x_{t+1})]. \end{aligned}$$

– We can ignore time subscript and write

$$V_j(x) = \max[f(x, u) + \beta V_{j-1}(x')],$$

where x' denotes next-period value.

2. Bellman Equation for Infinite Horizon DP

$$Under\,some\,conditions, starting\,from\,V_0\,which\,is$$
$$bounded\,and\,continuous,\ \lim_{j\to\infty} V_j = V.$$
$$Thus, the\,Bellman\,equation\,becomes$$
$$V(x) = \max_u \{\, f(x,u) + \beta V(x')\}$$
$$st.\ \ x' = g(x,u)\,and\,x\,is\,given.$$
$$Then, V(x_0)\,is\,the\,solution\,to\,(P1).$$

2. Bellman Equation for infinite horizon DP

The value function for problem (P1) satisfies the Bellman equation

$$V(x) = \max_{u \in U} \Big[f(x,u) + \beta V(g(x,u)) \Big]$$

$or, equivalently$

$$V(x) = \max_{u \in U} \Big[f(x,u) + \beta V(x') \Big]$$

$where\, x' = g(x,u)\, with\ x\, given.$

2. Bellman Equation for infinite horizon DP

- Bellman equation is a functional equation, in which the unknown function $V(x)$ appears on both sides.
- We seek a function V(x) that satisfy Bellman.
- Idea: we seek a fixed point of the operator.
- We regard RHS of Bellman as an operator (or a function of a function). Given V(x), RHS defines a new function of x. The solution is a function that leads to itself.
- Bellman equation for infinitely-horizon DP yields a uniqueness and existence of $V(x)$ by the contraction mapping theorem. (that is the fixed point exits and unique

2. Bellman Equation for infinite horizon DP

- Under some regularity conditions:

  (a) $V(x_0)$ is a unique strictly concave function.
  (b) Control $u(x)$ that max RHS of Bellman is the optimal control that is unique and time-invariant of the form $u_t = h(x_t)$. An agent uses time-invariant decision rule.

  (c) for interior solutions $V(x)$ is differentiable.
- One set of regularity conditions that works: f is concave, the constraint set is convex and compact.

2. Bellman Equation for infinite horizon DP

Notice that (a) there is no terminal condition we can use to derive the value function. Thus, we cannot find the optimal u by solving max problem on RHS of Bellman equation.

However, having $V(x)$ on both sides will provide us with a means of solving the functional equation.

(b) Time does not enter Bellman equation. We use the stationarity of the problem to make arguments about the existence of a value function that satisfies Bellman.

(c) What matters here is the size of the state variable at the beginning of the period.

3. Solving DP or Bellman Equation

- Generally, it is difficult to use the Bellman equation to find $V(x)$.
- Because we need to know something about the function in order to maximize the Bellman.
- We have two methods for solving Bellman.
- First method is value function iteration: starting from last period V back to the current until $V(x)$ has converged.
- The second method involves guessing a solution V and verify it.

3. Solving DP: value function iteration

- Now, we show an example by iterating on the Bellman equation or brute force technique to find $V(x)$ and $u(x)$.
- Let start with our cake-eating example.

$$Ex1. \qquad \max \sum_{t=0}^{\infty} \beta^t \ln(c_t),$$

$$s.t. \quad a_{t+1} = (1+r)a_t - c_t,$$

$$a_0 \text{ given.}$$

$$Bellman : V(a) = \max[\ln(c) + \beta V(a')]$$

$$st.\ a' = (1+r)a - c.$$

3. Solving DP: value function iteration

$$\text{Bellman}: V(a) = \max[\ln(c) + \beta V(a')]$$

$$st.\ a' = (1+r)a - c.$$

$$\text{Let start with } V_0(a) = \ln(a) \text{ as a starting point.} [\textit{last period}]$$

$$Next \qquad V_1(a) = \max[\ln(c) + \beta \ln((1+r)a - c)]$$

$$FOC\ is \qquad \frac{1}{c} - \frac{\beta}{a'} = 0 \Rightarrow a' = \beta c$$

$$\text{using } a' = (1+r)a - c: \qquad (1+r)a - c = \beta c,$$

$$thus \qquad c = h_1(a) = \left(\frac{1+r}{1+\beta}\right)a.$$

$$Sub \text{ in } V_1(a) = \ln(h_1(a)) + \beta \ln[(1+r)a - h_1(a)]$$

$$= A_1 + (1+\beta)\ln(a).$$

3. Solving DP: value function iteration

$$Next \qquad V_2(a) = \max[\ln(c) + \beta V_1(a')]$$

$$= \max[\ln(c) + \beta(A_1 + (1+\beta)\ln(a'))]$$

$$FOC\,is \qquad \frac{1}{c} - \frac{\beta(1+\beta)}{a'} = 0 \Rightarrow a' = \beta(1+\beta)c$$

$$u\sin g\, a' = (1+r)a - c: \qquad (1+r)a - c \;= \beta(1+\beta)c,$$

$$thus \qquad c = h_2(a) = \left(\frac{1+r}{1+\beta+\beta^2}\right)a.$$

$$Sub\,\mathrm{in}\,V_2(a) = \ln(h_2(a)) + \beta(A_1 + (1+\beta)\ln[(1+r)a - h_2(a)])$$

$$= A_2 + (1+\beta+\beta^2)\ln(a).$$

3. Solving DP: value function iteration

$$Continuing\ this\ process\ we\ will\ find$$

$$V_j(a) = A_j + (1 + \beta + \beta^2 + ... + \beta^j)\ln(a),$$

$$and\ c = h_j(a) = \left(\frac{1+r}{1 + \beta + \beta^2 + ... + \beta^j}\right)a.$$

$$If\ let\ j \to \infty, we\ get$$

$$V(a) = \lim V_j(a) = A\ + \frac{1}{1-\beta}\ln(a),$$

$$and \qquad c = \left(\frac{1+r}{1/1-\beta}\right)a = (1-\beta)(1+r)a.$$

$$sub\ in\ a'\ gives$$

$$a' = (1+r)a - (1+r)(1+\beta)a = \beta(1+r)a.$$

3. Solving DP: value function iteration

$$So, optimal\, policy\, functions\, are, for\, all\, t,$$

$$c_t = (1-\beta)(1+r)a_t,\ and$$

$$a_{t+1} = \beta(1+r)a_t.$$

$$Interpretation:$$

3. Solving DP: value function iteration

$\text{Homework. Brock and Mirman (1972): DMT ex.1.1}$

$$\max_{\{c_t, k_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \ln(c_t)$$

$$st. \qquad k_{t+1} = \bar{A} k_t^{\alpha} - c_t.$$

$$k_0 \ \ given \ (0 < \alpha, \beta < 1)$$

$$Start \ \ with \, V_0(k) \equiv 0 \, to \, show \, that$$

$$V(k) = \frac{1}{1-\beta}\left[\ln A(1-\alpha\beta) + \frac{\alpha\beta}{1-\alpha\beta}\ln A\beta\alpha\right]$$

$$+ \frac{\alpha}{1-\alpha\beta}\ln k.$$

3. Solving DP: value function iteration

$$Bellman: V(k) = \max[\ln(c) + \beta V(k')]$$

$$st.\ k' = \bar{A}k^{\alpha} - c.$$

$$Let\ start\ with\ V_0(k') = 0.$$

$$For\ one\ period\ problem:$$

$$V_1(k) = \max_{c,k'}[\ln(c) + \beta V_0(k')]$$

$$s.t.\quad c + k' = \bar{A}k^{\alpha}.$$

$$V_1(k) = \max_{k'}[\ln(\bar{A}k^{\alpha} - k') + 0]$$

$$\Rightarrow k' = 0; c = \bar{A}k^{\alpha}$$

$$\Rightarrow V_1(k) = [\ln(c)] = \ln(\bar{A}k^{\alpha}).$$

$$\Rightarrow V_1(k) = \ln\bar{A} + \alpha\ln k = A_1 + \alpha\ln k.$$

$$Two\ period\ problem: find\ V_2(k)$$

$$V_2(k) = \max_{c,k'}[\ln c + \beta V_1(k')] \quad st. \quad c + k' = \bar{A}k^{\alpha}$$

$$and \ V_1(k) = A_1 + \alpha \ln k.$$

$$V_2(k) = \max_{k'}[\ln(\bar{A}k^{\alpha} - k') + \beta(A_1 + \alpha \ln k')]$$

$$FOC: \frac{-1}{\bar{A}k^{\alpha} - k'} + \frac{\beta\alpha}{k'} = 0$$

$$\Rightarrow k' = \frac{\beta\alpha}{1+\alpha\beta}\bar{A}k^{\alpha}; \ c = \frac{1}{1+\alpha\beta}\bar{A}k^{\alpha}$$

$$V_2(k) = \ln[\frac{1}{1+\alpha\beta}\bar{A}k^{\alpha}] + \beta\{A_1 + \alpha\ln(\frac{\beta\alpha}{1+\alpha\beta}\bar{A}k^{\alpha})\}$$

$$= A_2 + \alpha(1+\alpha\beta)\ln k.$$

$$Three\ period: find\ V_3(k)$$

$$V_3(k) = \max_{c,k'}[\ln c + \beta V_2(k')] \quad st. \quad c + k' = \bar{A}k^{\alpha},$$

$$and\ V_2(k) = A_2 + \alpha(1+\alpha\beta)\ln k.$$

$$V_3(k) = \max_{k'}[\ln(\bar{A}k^{\alpha} - k') + \beta(A_2 + \alpha(1+\alpha\beta)\ln k')]$$

$$\text{from FOC, you can show that } k' = \frac{\beta\alpha + \beta^2\alpha^2}{1+\alpha\beta+\alpha^2\beta^2}\bar{A}k^{\alpha};$$

$$V_3(k) = A_3 + \alpha(1+\alpha\beta+\alpha^2\beta^2)\ln k, \text{ and so on...}$$

$$\text{Sequence of } B \text{ are } \quad \alpha, \alpha(1+\alpha\beta), \alpha(1+\alpha\beta+\alpha^2\beta^2), \ldots$$

$$Thus, \text{by taking limit}, V(k) = A + \frac{\alpha}{1-\alpha\beta}\ln k. \qquad \square$$

4. Solving DP: Guess and Verify method

- Now, we show another computational method for solving DP: guess and verify method.
- If the functional form of $V(x)$ is available, this method can save times.
- When f is CRRA, CARA, and quadratic, $V(x)$ takes the same general functional form. For example, if $f = ln(c)$, we could guess $V(x)$ is also logarithmic in x.
- We will make a function of the correct form, but leave its coefficient undetermined.

4. Solving DP: Guess and Verify method

- There are two special types of problems for which analytical solutions can be obtained from this method.
- It involves Cobb-Douglas constraint and log preference (as in our previous Brock and Mirman example), or linear constraint with quadratic preference. For example,

$$f = x_t^2 + u_t^2 \;\; and \, x_{t+1} = 2x_t + u_t.$$

- Now, let solve our cake-eating problem with Guess and verify method

$$\max_{\{c_t, k_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \ln(c_t)$$

$$st. \quad a_{t+1} = (1+r)a_t - c_t.$$

$$a_0 \ given.$$

From previous Example, we might guess the value function

$$V(a) = \phi + \theta \ln(a).$$

$$(1)\ Bellman: V(a) = \max[\ln(c) + \beta V(a')]$$

$$st.\ a' = (1+r)a - c.$$

$$(2)\ Use\ guess: V(a) = \max[\ln(c) + \beta\{\phi + \theta \ln(a')\}]$$

$$(3)\ Sub\ a': V(a) = \max[\ln(c) + \beta\{\phi + \theta \ln((1+r)a - c)\}$$

$$(4)\, FOC\, wrt\, c: \quad \frac{1}{c} - \frac{\beta\theta}{a'} = 0. \Rightarrow a' = \beta\theta c = (1+r)a - c$$

$$(5)\, find\, c: c = h(a) = \left(\frac{1+r}{1+\beta\theta}\right) a.\ (waiting\ for\ \theta)$$

$$(3')\, replace\, c: V(a) = \max[\ln((1+r)a - a') + \beta\{\phi + \theta \ln a'\}]$$

$$(4')\, FOC\, wrt\, a': \frac{-1}{(1+r)a - a'} + \frac{\beta\theta}{a'} = 0$$

$$[for\ R \equiv (1+r)] \qquad a' = \left(\frac{\beta\theta R}{1+\beta\theta}\right) a,\ \ wait\ for\ \theta.$$

$$(5')\, Sub\, c \text{ in } Bellman(2)\, or\ a'\, in\, (3')$$

$$V(a) = \ln\left[\frac{Ra}{1+\beta\theta}\right] + \beta\phi + \beta\theta \ln\left[Ra - \left(\frac{R}{1+\beta\theta}\right) a\right]$$

$$= \ln\left[\frac{Ra}{1+\beta\theta}\right] + \beta\phi + \beta\theta \ln\left[Ra - \left(\frac{R}{1+\beta\theta}\right)a\right]$$

$$= \ldots\ldots + (1+\beta\theta)\ln(a)$$

$compare\ with\ your\ guess, it\ holds\ if$

$$\phi + \theta\ln(a) = \ldots + (1+\beta\theta)\ln(a)$$

$$\Rightarrow \theta \qquad = (1+\beta\theta)\ or\ \theta = 1/(1-\beta)$$

$similary\ for\ \phi\ can\ be\ solved.$

$Thus, we\ just\ verify\ our\ guess\ that$

$$V(a) = \phi + \frac{1}{1-\beta}\ln(a).$$

$Use\ \theta\ in\ (5), we\ find\ c_t = h(a_t) = (1-\beta)(1+r)a_t$

$$Put\ c_t\ in\ a_{t+1}\ gives\ a_{t+1} = (1+r)a_t - (1-\beta)(1+r)a_t$$

$$= \beta(1+r)a_t. \qquad \square$$

4. Solving DP: Guess and Verify method

$Homework. From\ Brock\ and\ Mirman,\ let\ assume\ A = 1$
$and\ \alpha = 1\ in\ the\ state\ equation.$

$$\max_{\{c_t, k_t\}_{t=0}^{\infty}} \quad \sum_{t=0}^{\infty} \beta^t \ln(c_t)$$

$$st. \qquad k_{t+1} = k_t - c_t.$$

$$k_0 \ given\ (0 < \alpha, \beta < 1).$$

$From\ previous\ example, we\ know\ the\ answer\ is$

$$V(k) = (1-\beta)^{-1} \ \ln(1-\beta) + \tfrac{\beta}{1-\beta} \ln\beta \ + \ \tfrac{1}{1-\beta} \ \ln k.$$

4. Solving DP: Guess and Verify method

$$Guess\ V(k) = A + B\ln(k)\ \ for\ \ all\, k.$$

$$Given\, this\, known\, closed - form\, solution, we\, need$$

$$to\, find\, values\, of\ A\, and\, B\, to\, verify\, it.$$

$$A + B\ln(k) = \max_{k'}[\ln(k - k') + \beta(A + B\ln(k'))]$$

$$FOC: \frac{-1}{k - k'} + \frac{\beta B}{k'} = 0$$

$$k' = \frac{\beta B}{1 + \beta B}k. \qquad (wait\, for\, B)$$

$$Substituting\, k'\ \text{into the} \text{Bellman equation gives}$$

4. Solving DP: Guess and Verify method

$$A + B\ln(k) = \ln(k/(1+\beta B)) + \beta(A + B\ln(\frac{\beta B}{1+\beta B}k))$$

$$= ... + (1+\beta B)\ln(k).$$

$Compare\ both\ sides\ for\ B$

$$B = 1 + \beta B \Rightarrow B = \frac{1}{1-\beta}\ needed\ for\ a\ solution.$$

$Then\ use\ B\ to\ find\ A.\ (you\ can\ try\ this)$

$In\ all,\ we\ find\ (A,B)\ to\ verify\ our\ guess\ that$

$$V(k) = A + B\ln(k)\ holds\ with\ values\ defined\ above.$$

$With\ this\ solution,\ then\ we\ can\ find$

$$k_{t+1} = \frac{\beta B}{1+\beta B}k_t = \frac{\beta\frac{1}{1-\beta}}{1+\beta/(1-\beta)} = \beta k_t,\ and$$

$$from\ k_{t+1} = k - c_t;\ \beta k_t = k - c_t$$

$$c_t = (1-\beta)k_t.(as\ \text{your homework})\square$$

5. using Benveniste-Scheinkman

If we find $u_t = h(x)$, a time-invariant decision rule or policy function, which solves

$$V(x) = \max_u \{ f(x,u) + \beta V(g(x,u)) \}$$

Substitute into RHS. to get

$$V(x) = \max_u \{ f(x,h(x)) + \beta V(g(x,h(x))) \}$$

B-S formula: for V is differentiable,

$$V'(x) = \frac{\partial f[x,h(x)]}{\partial x} + \beta V'(g(x,h(x)))\left(\frac{\partial g(x,h(x))}{\partial x}\right)$$

5. using Benveniste-Scheinkman

If $\dfrac{\partial g(x,h(x))}{\partial x} = 0$, that is no state variable in RHS of state equation, $x_{t+1} = g_t(u_t)$. This can be made by redefining control. Then, B-S becomes

$$V'(x) = \frac{\partial f[x,h(x)]}{\partial x} + \beta V'(g(x,h(x)))\left(\frac{\partial g(x,h(x))}{\partial x}\right)$$

$$\boxed{V'(x) = \frac{\partial f[x,h(x)]}{\partial x}.}$$

For example : $A_{t+1} = R_t u_t \Rightarrow \dfrac{\partial A_{t+1}}{\partial A_t} = 0.$

5. using Benveniste-Scheinkman

This formula is useful for we need not to find V to get Euler equation.

Step 1. write down Bellman

Step 2. put transition equation in Bellman: all appear in terms of state.

Step 3. Find FOC w.r.t. $\mathit{control}\ u$.

Step 4. Find B-S. $[\mathit{Differentiate\ Bellman\ wrt}\ x]$ and

write B-S $\mathit{in\ terms\ of}\ u$.

Step 5. use B-S in FOC to find u.

Ex 5. Solves our cake-earting problem using B-S formula.

$$\max \sum_{t=0}^{\infty} \beta^t \ln(c_t),$$

$$s.t. \quad a_{t+1} = (1+r)a_t - c_t,$$

$$a_0 \text{ given.}$$

$Bellman:$

$$(1)\, write\, Bellman: V(a) = \max[\ln(c) + \beta V(a')]$$

$$st.\ a' = (1+r)a - c.$$

$$(2)\, Substitute: V(a) = \max[\ln(c) + \beta V((1+r)a - c)]$$

$$(3) FOC: \quad \frac{1}{c} + \beta V'(a')(-1) = 0 \qquad \Rightarrow V'(a') = 1 / c\beta$$

$$(4) BS\ from\ (2): \quad V'(a) = \beta V'(a')(1+r)$$

$$(5) useFOCinBS: \quad V'(a) = \frac{1}{c}(1+r)\, and\, \boxed{V'(a') = \frac{1}{c'}(1+r)}$$

$$Sub\, V'(a')\, in\, FOC: \qquad \frac{1}{c} = \beta \frac{1}{c'}(1+r) \qquad [Euler]$$

$$or \qquad c' = \beta(1+r)c$$

$$or \qquad c_{t+j} = [\beta(1+r)]^j c_t$$

$$[\text{remind}: \qquad u'(c) = \beta(1+r)u'(c')]$$

$To\, solve\, for\, optimal\, policy\, c_t = h(a_t), we\, use\, the$

$transition\, equation \quad a_{t+1} = (1+r)a_t - c_t.$

$$\text{Rewrite as} \quad a_t = (1+r)^{-1}(a_{t+1} + c_t) \qquad (*)$$

$We\, want\, to\, replace\, a_{t+1}\, in\, RHS\, of\, (*)\, by\, repeating$

$substitution forward.\; Putting\, one-step\, ahead\, in\, RHS\, by$

$$a_t = (1+r)^{-1}[(1+r)^{-1}(a_{t+2} + c_{t+1}) + c_t]$$

$$= (1+r)^{-1}[(1+r)^{-1}(a_{t+2} + c_{t+1}) + c_t]$$

$$= \frac{a_{t+2}}{(1+r)^2} + \frac{c_{t+1}}{(1+r)^2} + \frac{c_t}{(1+r)}$$

$$= \frac{a_{t+2}}{(1+r)^2} + \frac{1}{(1+r)}\left(\frac{c_{t+1}}{(1+r)} + c_t\right)$$

$$= \frac{a_{t+2}}{(1+r)^2} + \frac{1}{(1+r)}\left(\sum_{j=0}^{1}\left(\frac{1}{1+r}\right)^j c_{t+j}\right)$$

$$thus\ for\ (T-1) step\ ahead, we\ get$$

$$a_t = \frac{a_{t+T}}{(1+r)^T} + \frac{1}{(1+r)}\left(\sum_{j=0}^{T-1}\left(\frac{1}{1+r}\right)^j c_{t+j}\right)$$

$$Till\, t = T, we\ get$$

$$a_t = \frac{1}{1+r}^{T} a_{t+T} + \frac{1}{1+r} \sum_{j=0}^{T-1} \left(\frac{1}{1+r}\right)^j c_{t+j}$$

$$\text{Take limit, for } \lim_{T\to\infty} \frac{1}{1+r}^{T} a_{t+T} = 0, we\, have$$

$$a_t = 0 + \frac{1}{1+r} \sum_{j=0}^{\infty} \left(\frac{1}{1+r}\right)^j c_{t+j}.$$

$$Thus, \quad (1+r)a_t = \sum_{j=0}^{\infty} \left(\frac{1}{1+r}\right)^j c_{t+j}$$

$$Sub\, c_{t+j}\ from\ Euler, we\ get$$

$$(1+r)a_t = \sum_{j=0}^{\infty} \left(\frac{1}{1+r}\right)^j [\beta(1+r)]^j c_t = \sum_{j=0}^{\infty} \beta^j c_t.$$

$$(1+r)a_t = \sum_{j=0}^{\infty} \beta^j c_t.$$

$$For \sum_{j=0}^{\infty} \beta^j = 1 + \beta + \beta^2 + ... = \frac{1}{1-\beta},$$

$$thus \qquad c_t = (1-\beta)(1+r)a_t,$$

$$and \qquad a_{t+1} = \beta(1+r)a_t. \qquad \square$$

5. using Benveniste-Scheinkman

Homework. Solves Brock and Mirman *again*,

using B-S formula in guess and verify, see p.26 in DMT.

$$\max_{\{c_t, k_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \ln(c_t)$$

$$st. \qquad k_{t+1} = \bar{A} k_t^{\alpha} - c_t.$$

$$k_0 \ \ given \ (0 < \alpha, \beta < 1)$$

$$Guess: V(k) = E + F \ln(k)$$

$$Bellman: V(k) = \max_{k'}\{\ln(\bar{A}k^{\alpha} - k') + \beta V(k')\}$$

$$= \max_{k'}\{\ln(\bar{A}k^{\alpha} - k') + \beta(E + F\ln(k')\}$$

$$FOC: \frac{-1}{\bar{A}k^{\alpha} - k'} + \beta\frac{F}{k'} = 0 \Rightarrow k' = \frac{\beta F}{1+\beta F}\bar{A}k^{\alpha}$$

$$B - S: V'(k) = \frac{1}{\bar{A}k^{\alpha} - k'}\alpha\bar{A}k^{\alpha-1}$$

$$***Use\, k'\, in\, B-S\, gives$$

$$V'(k) = \frac{1}{\bar{A}k^{\alpha} - \frac{\beta F}{1+\beta F}\bar{A}k^{\alpha}}\alpha\bar{A}k^{\alpha-1}$$

$$= (1+\beta F)\alpha k^{-1}$$

$$From\, our\, guess, \qquad V'(k) = Fk^{-1}$$

$$thus \qquad (1+\beta F)\alpha k^{-1} = Fk^{-1}$$

$$F = \alpha/(1-\alpha\beta)$$

$$\Rightarrow V(k) = E + \frac{\alpha}{(1-\alpha\beta)}\ln(k)$$

$$and \quad k' = \frac{\beta F}{1+\beta F}\bar{A}k^{\alpha} = A\alpha\beta k^{\alpha}$$