Example 7.C: Determine all the optimizers of

$$f(x, y) = x^3 - y^3 - 9xy$$

1-step: Find $(x^*, y^*) \Rightarrow f_x = 0$, $f_y = 0$

$$\begin{cases} f_x = 3x^2 - 9 \cdot y = 0 \\ f_y = -3y^2 - 9x = 0 \end{cases}$$

$f_x \Rightarrow 3x^2 = 9 \cdot y \Rightarrow x^2 = 3 \cdot y$

$f_y \Rightarrow 3y^2 = -9 \cdot x \Rightarrow y^2 = -3 \cdot x$

$x = -\frac{1}{3} \cdot y^2$

$$\left(-\frac{1}{3} y^2\right)^2 = 3 \cdot y$$

$$y^4 = 27 \cdot y$$

$$y^4 - 27 \cdot y = y(y^3 - 27) = 0$$

$y = 0$, $x = 0$ $\qquad$ $y = 3$, $x = -3$

Two pairs of (x, y) that satisfying the two equations from the F.O.C.

(i) $(x, y) = (0, 0)$

(ii) $(x, y) = (-3, 3)$

Example 7.D: Determine all the optimizers of

$$f(x,y) = x^2 - 2xy + 3y^2 + 2x - 2y$$

Step 1 : Find (x^*, y^*) => $f_x = 0$

$f_y = 0$

$f_x = 2x - 2y + 2 = 0$

$f_y = -2x + 6y - 2 = 0$

→ f_x => $x = y - \cancel{2}\ 1$

Substitute $x = y - \cancel{2}\ 1$ to f_y

f_y => $-2(y - \cancel{2}\ 1) + 6y - 2 = 0$

$4y \quad \cancel{4y + 2} = 0$

$x = -1,\ y = 0$ $\quad y = \cancel{-\frac{1}{2}}\ 0$

$x = \cancel{-\frac{5}{2}}\ -1$

$$H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ -2 & 6 \end{bmatrix}$$

$x = -1, y = 0$

$|H_1| = 2 > 0$; $|H_2| = 12 - (-2)(-2) = 8 > 0$

→ H is positive definite → $d^2f(-1,0)$ is convex

→ Convex at $(-1,0)$ suggests that the point is a local minimum point

2-step testing for Concavitity / Convexity

$f_x = 3x^2 - 9y$
$f_y = -3y^2 - 9x$

$$H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} 6x & -9 \\ -9 & -6y \end{bmatrix}$$

(i) $(x,y) = (0,0)$ → unfortunitely, we dont know.

$$H\big|_{(x,y)=(0,0)} = \begin{bmatrix} 6(0) & -9 \\ -9 & -6(0) \end{bmatrix} = \begin{bmatrix} 0 & -9 \\ -9 & 0 \end{bmatrix}$$

* $|H_1| = 0$; $|H_2| = -81$ Neither positive nor negative definite

(ii) $(x,y) = (-3,3)$

$$H_{(x,y)=(-3,3)} = \begin{bmatrix} -18 & -9 \\ -9 & -18 \end{bmatrix}$$

heassien metrix is not definiteness

$|H_1| = -18$; $|H_2| = (-18)^2 - (-9)^2 = 243$

H is negative definite at $(x,y) = (-3,3)$
$d^2f(-3,3)$ is less then zero → f is concave at $(x,y) = (-3,3)$.

As as result we conclude that $(x,y) = (-3,3)$ is the maximum point.

If F^2 is global concave (convex) → local max (min) is ~~warranted~~ gauranteed to be global max (min).

Definition: *Globally concave/convex*

A function is said to be globally concave (convex) if the function is concave (convex) for the entire domain set defining the function.

Following the theorem of definiteness on matrix, we can check for the global property of a function.

$\forall (x,y) \in f$; f concave if $d^2f < 0$
f convex if $d^2f > 0$

Lemma:

- Globally concave:
  - Hessian matrix is negative definite for all (x,y).
- Globally convex
  - Hessian matrix is positive definite for all (x,y).

Example: $f(x,y) = e^{(x+y)}$, $\quad g(x,y) = -(x^2 + y^2)$

$$H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$$

$f_x = e^{x+y}$

$f_y = e^{x+y}$

$f_{xx} = e^{x+y}$

$f_{yy} = e^{x+y}$

$g(x,y)$

$g_x = -2x$

$g_y = -2y$

$g_{xy} = g_{yx} = \cancel{-2}\ 0$

$$H = \begin{bmatrix} -2 & \cancel{-2}\ 0 \\ \cancel{-2}\ 0 & -2 \end{bmatrix}$$

$\forall (x,y)$ → global concavity

$|H_1| = -2$

$|H_2| = 4$

$\Big\}$ H is always negative definite at any (x,y) defining your function.

Why do we care about this?

- Optimized solution in economics attempt at searching for the first-best, i.e. global solution.
- Unfortunately, mathematical optimization only provides toolkits for searching/identifying local optimizers.

Theorem:

If a function is globally convex (concave), local minimizer (maximizer) is warranted to be global minimizer (maximizer).

7.2.3) Some economics applications

a) Basic profit maximization and Factor inputs decision

- **Structure of the problem**

  a. Firm owns a technology that requires the use of some types of factor inputs.

  b. Technology can be represented by a production function.

     i. For simplicity, I will assume only two types of the factor inputs, capital (K) and labor (L)

     ii. Capital costs firm "r" per unit of installation, and labor costs firm "w" per unit of labor usage.

  c. Firm sells the product to the market, getting back "p" in return, per unit of output sold.

  d. Question under the problem is that "*how much to utilize each type of factor input to get the highest level of profit?*"

Objective function:

Choice variables:

Optimality conditions: FOC

Second-order condition

- Hessian of profit function is

$$H = \begin{bmatrix} \pi_{KK} & \pi_{KL} \\ \pi_{LK} & \pi_{LL} \end{bmatrix} = \begin{bmatrix} pf_{KK} & pf_{KL} \\ pf_{LK} & pf_{LL} \end{bmatrix} = p \begin{bmatrix} f_{KK} & f_{KL} \\ f_{LK} & f_{LL} \end{bmatrix}$$

- So, we need $\begin{bmatrix} f_{KK} & f_{KL} \\ f_{LK} & f_{LL} \end{bmatrix}$ to be negative definite because $p > 0$.
  - This condition ensures the solution from FOC as a maximizer.
  - What does it mean in economics for having negative definitive in $\begin{bmatrix} f_{KK} & f_{KL} \\ f_{LK} & f_{LL} \end{bmatrix}$?
  - What could it be implied about the mathematical property of the production function?

N-variable case

Objective function: $z = f(x_1, x_2, x_3, \dots, x_n)$

FOC. Step 1

$f_1 = f_2 = f_3 = \cdots = f_n = 0$ ➔ solve for the solution from FOCs.

↳ n equations } solve.
n unknown

SOC. Step 2

$$H = \begin{bmatrix} f_{11} & f_{12} & \cdots & f_{1n} \\ f_{21} & f_{22} & \cdots & f_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ f_{n1} & f_{n2} & \cdots & f_{nn} \end{bmatrix}$$

"alternate sign"

- H is negative definite ➔
  - $|H_1| < 0, |H_2| > 0, |H_3| < 0$ ....
  - sign($|H_i|$) = $(-1)^i$ for each "i".
  - Local ~~min~~ max

$d^2f < 0$

f: concave

i = odd → −

i = even → +

- H is positive definite ➔
  - $|H_1| > 0, |H_2| > 0, |H_3| > 0$ ....
  - sign($|H_i|$) > 0 for *all* "i".
  - Local ~~max~~ min

$d^2f > 0$

f: convex

marginal profit of labor

marginal profit of Capital

Perfectly Comp. Market.

Example 7.E Suppose $Q = K^{\frac{1}{2}} + L^{\frac{1}{2}}$ and price is fixed, equal to P. Assume that "w" as wages and "r" as rental.

$$\frac{\partial \pi}{\partial K} = P \cdot \left(\tfrac{1}{2} K^{-\frac{1}{2}}\right) - r$$

a) Setting up the profit function.

$$\frac{\partial \pi}{\partial L} = P\left(\tfrac{1}{2} L^{-1/2}\right) - w$$

$$\pi = TR - TC$$

$$\pi(K, L) = \underbrace{P \cdot \left(K^{\frac{1}{2}} + L^{\frac{1}{2}}\right)}_{P \cdot Q} - (\underbrace{w \cdot L + r \cdot K}_{TC})$$

b) Derive the optimal level of capital installation and labor usage.

objective firm: (K^*, L^*) that maximizes $\pi(K, L)$

$K^* \rightarrow$ optimal level of capital installation

$L^* \rightarrow$ optimal level of labor usage (employment)

Apply the two-step procedure:

Step 1 : $\frac{\partial \pi}{\partial K} = 0 = P\left(\tfrac{1}{2} K^{-\frac{1}{2}}\right) - r$

$\frac{\partial \pi}{\partial L} = 0 = P\left(\tfrac{1}{2} L^{-\frac{1}{2}}\right) - w$

$P\left(\tfrac{1}{2} K^{-\frac{1}{2}}\right) = r$ / $P\left(\tfrac{1}{2} L^{-\frac{1}{2}}\right) = w$

$VMP_K = P \cdot MP_K$ Unit Cost of hiring

~~a more un~~

one more unit of Capital

(i) $P\left(\frac{1}{2}K^{-\frac{1}{2}}\right) = r$

(ii) $P\left(\frac{1}{2}L^{-\frac{1}{2}}\right) = W$

Speci'l Case

No Cross-price Effect.

$$\boxed{\left.\begin{aligned} K^* &= \left(\frac{P}{2r}\right)^2 = \frac{P^2}{4} r^{-2} \\ L^* &= \left(\frac{P}{2w}\right)^2 = \frac{P^2}{4} \cdot W^{-2} \end{aligned}\right\}}$$

Demand for Capital

$\longrightarrow$ Sol^n the profit max problem.

Demand for Labor.

$$\frac{\partial K^*}{\partial P} = 2\left(\frac{P}{2r}\right) \cdot \left(\frac{1}{2r}\right) > 0$$

$P \cdot MP_K = r$

$\therefore$ if $P\uparrow$ the initial level of K would no longer be optimal.

$$\frac{\partial L^*}{\partial P} = 2\left(\frac{P}{2W}\right)\left(\frac{1}{2W}\right) > 0$$

$$\frac{\partial K^*}{\partial r} = \frac{P^2}{4}(-2r^{-3}) < 0 \left(\text{"}r\uparrow; K^* \text{ drops"}\right)$$

$$\frac{\partial L^*}{\partial w} = \frac{P^2}{4}(-2W^{-3}) < 0$$

$K^* = \left(\frac{P}{2r}\right)^2 > 0 \; ; \; L^* = \left(\frac{P}{2w}\right)^2 > 0$ $\quad$ $K > 0, L > 0$

suppose $P, w, r >> 0$

c) Check the second-order condition to confirm that your answer is correct.

- Check for property of $\pi(K, L)$ at K^*, L^*
- check for Definiteness of the hessian matrix of the profit f^n

$$H = \begin{bmatrix} P(-\frac{1}{4}K^{-\frac{3}{2}}) & 0 \\ 0 & P(-\frac{1}{4}L^{-\frac{3}{2}}) \end{bmatrix}$$

$|H_1| = P(-\frac{1}{4}K^{-\frac{3}{2}})$

- $|H_1|$ negative < 0 $\forall K > 0$

$|H_2| = |H|$

$P(-\frac{1}{4}K^{-3/2})(P(-\frac{1}{4}L^{-\frac{3}{2}}))$

$= P^2 \cdot \frac{1}{16} K^{-3/2} L^{-3/2}$

$|H_2| > 0$ > 0

$\forall K$ and $\forall L > 0$

"Alternate sign" ⇒ H is "negative definite"

That implies d^2f is less than zero.

f is concave f^n (globally)

Conclude: $K^*, L^* \rightarrow$ profit-maximizing level of inputs (global sol^n)

d) Calculate the level of maximized (optimal) profit.

e) What is the optimal level of capital installation and labor usage when P = 1000, w = 20 and r = 10.

Calculate π^*, K^* and L^* using these numbers

Exercise 7.A

Redo the **example 7.E**, but using the following Cobb-Douglas function: $Q = K^2L^3$ as the production technology of the firm.

- Derive the demand function for both capital and labor.
- Discuss some important properties of demand function of capital and labor.
- Derive the optimal profit function, and discuss some properties of the function.

$$\Pi(K,L) = P(K^{\frac{1}{2}} + L^{\frac{1}{2}}) - (w \cdot L + rK)$$

$$K^* = \left(\frac{P}{2r}\right)^2 \quad L^* = \left(\frac{P}{2w}\right)^2$$

maximized level of profit

$$\Pi(K^*, L^*) = P\left(\frac{P}{2r} + \frac{P}{2w}\right) - w \cdot \frac{P^2}{4w^2} - r \cdot \frac{P^2}{4r^2}$$

$$= \frac{P^2}{2r} + \frac{P^2}{2w} - \frac{P^2}{4w} - \frac{P^2}{4r}$$

$$= \frac{P^2}{4r} + \frac{P^2}{4w} = \Pi^*(w, r, P).$$

is the optimized profit function

$\boxed{\text{optimized-value profit function.}}$

optimal profit function.

b) Multi-plant problem Firm

Structure of the problem:

- Firm operates in a market, and sells only a single type of product in the market.

- Firm has several plants that can be used to fulfill the production. (Think about plants that are located in different cities, for example. Or, two plants in the same location, but each differs in their production technique.)

Q $Q_1 = ?$ $Q_2 = ?$

- How to optimality choose the capacity of each plant, given a certain level of the total output that firm attempts to sell to the market.

For simplicity, consider a 2-plan example.

- $C^1 = C^1(Q_1) = cost\ of\ plant\#\ 1,\ when\ firm\ chooses\ to\ produce\ Q_1$

- $C^2 = C^2(Q_2) = cost\ of\ plant\#\ 2,\ when\ firm\ chooses\ to\ produce\ Q_1$

The total production of these two plants must fulfil the total output (Q) that firm is trying to sell to the market. That is, $Q_1 + Q_2 = Q$

First, consider a simple case where firm is a price taker. That is, it treats price (p) as given.

objective is to maximize profit by choosing different level of production capacity chosen by each plant.

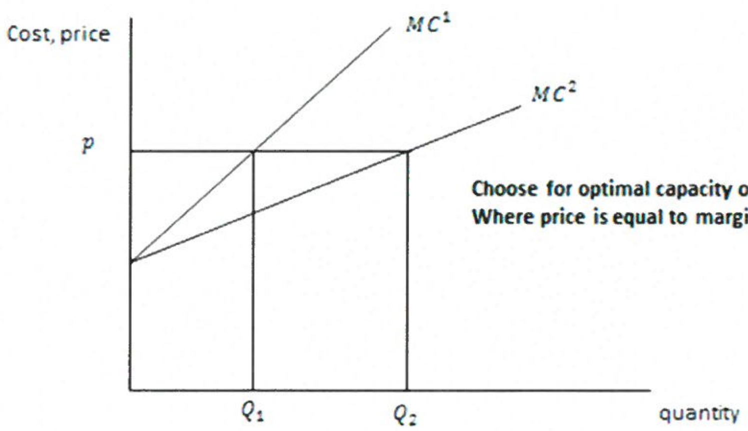
Optimality conditions:

$$\pi = P \cdot Q - C^1(Q_1) - C^2(Q_2)$$

$$\pi(Q_1, Q_2) = P(Q_1 + Q_2) - C^1(Q_1) - C^2(Q_2)$$

$$\left.\begin{aligned} \frac{\partial \pi}{\partial Q_1} &= P - MC^1(Q_1) = 0 \\ \frac{\partial \pi}{\partial Q_2} &= P - MC^2(Q_2) = 0 \end{aligned}\right\} \underline{MC^1 = MC^2 = P}$$

Setting up mathematical problem: Graphical



Choose for optimal capacity of each plant
Where price is equal to marginal of each plant