Solving Inequalities: I

TU152: Fundamental Mathematics

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Solving Inequities

1. **Solving Inequities**
2. **Solving Inequities with absolute values**

Inequities

We are interested in finding the solution set of x for each of the following inequalities

$$f(x) < 0, \qquad f(x) > 0, \qquad f(x) \leq 0, \qquad f(x) \geq 0$$

where $f(x)$ is a function in one these forms:

1. $f(x)$ is a polynomial

$$f(x) = a_0 + a_1 x + a_2 x + \cdots + a_n x^n$$

   where $a_0, a_1, \ldots, a_n$ are some constants and n is a positive integer.

2. $f(x)$ is a rational function, i.e. $f(x)$ is in the form:

$$f(x) = \frac{P(x)}{Q(x)}$$

   where $P(x)$ and $Q(x)$ are polynomials, $Q(x) \not\equiv 0$.

Properties of Inequities

Properties

(1) If $a < b$, then $a + c < b + c$ and $a - c < b - c$.

(2) If $a < b$ and $c > 0$ then $ac < bc$.
But if $a < b$ and $c < 0$ then $ac > bc$.
Note that mulitplication by a negative number flips the inequality.

(3) If $a < b$ and $b < c$, then $a < c$.

(4) If $a > 0$, then $\frac{1}{a} > 0$.

(5) If $0 < a < b$, then $\frac{1}{b} < \frac{1}{a}$

(6) If $0 < a < b$, then $\frac{1}{b^n} < \frac{1}{a^n}$ for any positive integer n.

(7) If $0 \leq a < b$, then $a^n < b^n$ for any positive integer n.

(8) If $0 < a < b$ and $0 < c < d$, then $0 < ac < bd$.

Inequality with quadratic polynomial

Consider the inequality involves quadratic polynomial

$$f(x) = ax^2 + bx + c,$$

for some constants $a \neq 0$, b, c.

- If $b^2 - 4ac \geq 0$ it is useful to use the following factorization:

$$ax^2 + bx + c = a(x - K_1)(x - K_2),$$

$$\boxed{K_1 = \left(\frac{-b + \sqrt{b^2 - 4ac}}{2a}\right) K_2 = \left(\frac{-b - \sqrt{b^2 - 4ac}}{2a}\right)}$$

- If $b^2 - 4ac < 0$, we cannot factor the polynomial in terms of real numbers. Moreover,
  - if $a > 0$, then we always have $ax^2 + bx + c > 0, \forall x \in \mathbb{R}$.
  - if $a < 0$, then we always have $ax^2 + bx + c < 0, \forall x \in \mathbb{R}$.

Proof of the case $b^2 - 4ac < 0$

Examples

Example:Find the solution set for each of the following inequalities.

1. $4x^4 - x^5 < 3x^3$
2. $x^3 - 7x > 36$

Inequality with rational function

Consider inequality with rational function $f(x)$:

$$f(x) < 0, \qquad f(x) > 0, \qquad f(x) \leq 0, \qquad f(x) \geq 0$$

where

$$f(x) = \frac{P(x)}{Q(x)},$$

$P(x)$ and $Q(x)$ are polynomials.
The solution set can be found by using a similar technique as in the case of polynomial inequalities. However, because rational expressions have denominators (and therefore may have places where they're not defined), you have to be a little more careful in finding your solutions.

1. Factor $P(x)$ and $Q(x)$ and find the roots for each of them.
   - The roots of $P(x)$ are the **zeros** of the rational function $f(x)$.
   - The roots of $Q(x)$ are the **undefined points** of $f(x)$.
2. Use these **zeroes** and **undefined points** to divide the number line into intervals.
3. Determine the sign of the rational on each interval.

Example: Find the solution set for following inequality.

$$\frac{1}{x+4} < \frac{1}{x}$$

Example: Find the solution set for following inequality.

$$\frac{x^3 - x^2}{x + 1} + 2x > 0$$

Example

Example: Find the solution set for following inequality.

$$\frac{x^2}{-x^2 + x - 4} \leq -1$$

Example: Find the solution set for following inequality.

$$\frac{20x^2 - 17x - 63}{e^x(x^2+1)} < 0$$

Example: Find the value of a so that $x \in (-\infty, -3/2] \cup [4, \infty)$ satisfies the following inequality:

$$-2x^2 + \frac{5}{2}xa + 3a^2 \leq 0.$$

Example

Example: Find the solution set for following inequality.

$$x^5 + 3x^4 - 23x^3 - 51x^2 + 94x + 120 \geq 0$$

Example