

## Solution: Assignment 2 (Part 2)

1. Determine the truth value of each of these statements. Explain your answer.

- (a)  $\forall n \in \mathbb{Z}, n^3 > n$       (b)  $\exists n \in \mathbb{Z}, n^2 + n = 1$       (c)  $\exists n \in \mathbb{R}, n^2 + n = 1$   
 (d)  $\forall n \in \mathbb{R}^+, 2n > \frac{1}{n}$       (e)  $\forall n \in \mathbb{R}, (n > 0) \rightarrow (3n > n)$

**Answer:**

- (a)  $\forall n \in \mathbb{Z}, n^3 > n$

**False** because we can find a counterexample  $n = -3$ .

$$n^3 = -27 \not> -3 = n.$$

Note any negative integer could be used as a counterexample.

- (b)  $\exists n \in \mathbb{Z}, n^2 + n = 1$

**False.** Note that  $n^2 + n = 1$  is true only when

$$n^2 + n - 1 = 0 \quad \text{or} \quad n = \frac{-1 \pm \sqrt{1 - 4(1)(-1)}}{2} = \frac{-1 \pm \sqrt{5}}{2}.$$

That is,  $n^2 + n - 1 = 0$  when  $n = \frac{-1 + \sqrt{5}}{2}$  or  $n = \frac{-1 - \sqrt{5}}{2}$ , which are not in the set of integers  $\mathbb{Z}$ . Therefore, there is no value in  $\mathbb{Z}$  that makes  $n^2 + n = 1$  true.

- (c)  $\exists n \in \mathbb{R}, n^2 + n = 1$

**True.** From (b), we see that  $n^2 + n - 1 = 0$  when  $n = \frac{-1 + \sqrt{5}}{2}$  or  $n = \frac{-1 - \sqrt{5}}{2}$ , which are in the set of real numbers  $\mathbb{R}$ . Hence the statement is true.

- (d)  $\forall n \in \mathbb{R}^+, 2n > \frac{1}{n}$

**False** because we can find a counterexample  $n = 1/2$ :  $2n = 1$  and  $\frac{1}{n} = 2$  and  $2n \not> \frac{1}{n}$ , or

$$2n = 2 \cdot \frac{1}{2} = 1 \not> 2 = \frac{1}{1/2} = \frac{1}{n}.$$

Note any positive rational number that is less than 1 could be used as a counterexample, e.g  $n = 1/3, 2/3, 1/5$ .

- (e)  $\forall n \in \mathbb{R}, (n > 0) \rightarrow (3n > n)$

**True.** To show that this universal statement is true we need to consider all possible value of  $n$  in  $\mathbb{R}$ . Let  $P(n)$  be  $R(n) \rightarrow S(n)$ , where  $R(n) = (n > 0)$  and  $S(n) = (3n > n)$ . When  $n > 0$ , then we have

- (i) For  $n > 0$ ,  $R(n)$  is true and  $S(n) = (3n > n) \equiv 3 > 1$  (by dividing both side by  $n > 0$ ), which is always true, i.e. " $T \rightarrow T$ "

Hence,  $P(n)$  is true.

- (ii) For  $n \leq 0$ ,  $R(n)$  is false and  $S(n) = (3n > n)$  is false, i.e. " $F \rightarrow F$ "

Hence,  $P(n)$  is (vacuously) true.

Also notice that

$$\begin{aligned} 3n &> n \\ 3n - n &> 0 \\ 2n &> 0 \\ n &> 0. \end{aligned}$$

That is,  $3n > n$  in equivalent to  $n > 0$ .

2. Let  $\mathbb{R}$  be the domain of  $x$ . Determine the **truth set** for each of these statements.

(a)  $P(x) : "x < \frac{1}{x}"$       (b)  $P(x) : "2x + 1 < 0 \text{ or } x \geq 1"$

**Answer:**

(a)  $P(x) : "x < \frac{1}{x}"$

$$x < \frac{1}{x} \iff x - \frac{1}{x} < 0 \iff \frac{x^2 - 1}{x} < 0$$

which can occur when either one of the following conditions happen.

(i) when  $x^2 - 1 < 0$  and  $x > 0$ , i.e.

$$-1 < x < 1 \text{ and } x > 0, \text{ or } x \in (-1, 1) \cap (0, \infty) = (0, 1)$$

(ii) when  $x^2 - 1 > 0$  and  $x < 0$ , i.e.

$$(x < -1 \text{ or } x > 1) \text{ and } x < 0, \text{ or } x \in \{(-\infty, -1) \cup (1, \infty)\} \cap (-\infty, 0) = (-\infty, -1).$$

Since the truth set will contain all the possible values of  $x \in \mathbb{R}$  that makes  $P(x)$  true, we have from (i) and (ii),  $P(x)$  is true when  $x \in (0, 1) \cup (-\infty, -1)$  the truth set is  $(-\infty, -1) \cup (0, 1)$ .

(b)  $P(x) : "2x + 1 < 0 \text{ or } x \geq 1"$

Since  $2x + 1 < 0 \iff x < -\frac{1}{2}$ ,  $x \in (-\infty, \frac{1}{2})$ . That is,  $P(x)$  is true either when  $x \in (-\infty, \frac{1}{2})$  or  $x \in [1, \infty)$ . Therefore, the truth set is  $(-\infty, \frac{1}{2}) \cup [1, \infty)$ .

3. Let the domain for variables  $x$  and  $y$  be the set of real numbers  $\mathbb{R}$ . Determine the truth values of the following statements. Explain your answer.

(a)  $\exists y \forall x, xy = x$       (b)  $\forall x \exists y, xy = x$       (c)  $\forall y \exists x, y = x$       (d)  $\exists x \forall y, y = x$

**Answer:**

(a) **True.** The statement  $\exists y \forall x, xy = x$  is **true** since we can use  $y = 1 \in \mathbb{R}$ , then we have  $x = x$  which is always true for all  $x \in \mathbb{R}$

(b) **True.** Consider  $\forall x \exists y, xy = x$ . To show that this statement is true, we need to consider all possible values for  $x \in \mathbb{R}$ , which will be dividing into two cases:  $x \neq 0$  and  $x = 0$ .

(i) For  $x \neq 0$ , " $xy = x$ " gives  $y = 1$  (by dividing both sides by  $x$ ). That is, we have that for any  $x \in \mathbb{R}$  with  $x \neq 0$ , there is  $y = 1 \in \mathbb{R}$  such that  $xy = x$ .

(ii) For  $x = 0$ , " $xy = x$ " gives  $0y = 0$ , which implies that  $y$  can be any real number to have this statement true (and hence there is at least one value of  $y$ ). That is, we have that for any  $x \in \mathbb{R}$  with  $x \neq 0$ , there is  $y = 1 \in \mathbb{R}$  such that  $xy = x$ .

From cases (i) and (ii) we have that for any given  $x \in \mathbb{R}$ , there is  $y \in \mathbb{R}$  such that  $xy = x$ .

(c)  $\forall y \exists x, y = x$

**True.** Given any fixed value of  $y \in \mathbb{R}$ , we can find  $x \in \mathbb{R}$  such that

$$y = x$$

by setting  $x = y$ . Therefore this is true.

(d)  $\exists x \forall y, y = x$

**False.** Suppose we fix a value of  $x \in \mathbb{R}$ . Let  $y = x + 1$ . So there exists  $y = x + 1 \in \mathbb{R}$  such that

$$y \neq x \quad (\text{since } y = x + 1 \neq x)$$

for any given  $x \in \mathbb{R}$ . Therefore, we cannot find  $x \in \mathbb{R}$  such that every value of  $y \in \mathbb{R}$  makes  $x = y$ .

4. Let  $Q(x, y, z)$  be the statement “ $xy = z$ .” If the domain for variables  $x, y, z$  is the set of all integers, determine the truth values of the following statements. Explain your answer.
- (a)  $Q(1, 2, 2)$       (b)  $Q(2, 0, 2)$       (c)  $\exists y, Q(2, y, 1)$   
 (d)  $\forall x \forall y \exists z, Q(x, y, z)$       (e)  $\exists z \forall x \forall y, Q(x, y, z)$

**Answer:**

- (a)  $Q(1, 2, 2)$   
 $Q(1, 2, 2) : 1 \cdot 2 = 2$  is **true**.
- (b)  $Q(2, 0, 2)$   
 $Q(2, 0, 2) : 2 \cdot 0 = 2$  is **false**.
- (c)  $\exists y, Q(2, y, 1)$   
 $\exists y \in \mathbb{Z}, 2y = 1$  is **false** because the equation  $2y = 1$  is true only when  $y = \frac{1}{2}$ , which is **not** in the set of integers  $\mathbb{Z}$ .
- (d)  $\forall x \forall y \exists z, Q(x, y, z)$  is **true**.  
 Given any fixed  $x, y \in \mathbb{Z}$ , we can set  $z = xy$  to make this statement true.
- (e)  $\exists z \forall x \forall y, Q(x, y, z)$  is **false**.  
 Given any fixed  $z \in \mathbb{Z}$ , it is impossible to have  $xy = z$ , for all  $x, y \in \mathbb{Z}$ .  
 To show that this statement is not true, we can show that its negation :

$$\sim (\exists z \forall x \forall y, z = xy) \equiv \forall z \exists x \exists y, z \neq xy$$

is true. We need to consider all possible values of integers  $z$ . Here we will divide the values of  $z \in \mathbb{Z}$  into 3 cases:

(i) For  $z = 1$ , we can set  $x = 0 \in \mathbb{Z}$  and  $y = 0 \in \mathbb{Z}$ , so we have  $xy = 0$  and  $z \neq xy$ .

(ii) For  $z = 0$ , we can set  $x = 1$  and  $y = 1$  so that  $z \neq xy$ .

(iii) For any fixed value  $z \in \mathbb{Z} - \{0, 1\}$ , we can set  $x = z \in \mathbb{Z}$  and  $y = z \in \mathbb{Z}$ , so that we have  $xy = z^2$ , which implies  $z^2 = z$ . Note that  $z^2 = z$  is true only when  $z = 0$  or  $z = 1$ , which are not included in this case. Therefore, for  $z \in \mathbb{Z} - \{0, 1\}$ , we can find  $x \in \mathbb{Z}$  and  $y \in \mathbb{Z}$  such that  $z \neq xy$ .

From above, we have shown that its negation is true and therefore the statement itself is **false**.

5. Let  $\mathbb{Z}^+$  be the domain of  $x$ . Let  $P(x)$  and  $Q(x)$  be the predicates “ $x$  is not divisible by 3,” and “ $x$  is divisible by 12,” respectively. Determine whether the following statements are true or false. Give a counterexample for each false statement.
- (a)  $Q(x) \Rightarrow P(x)$       (b)  $P(x) \Rightarrow \sim Q(x)$

**Answer:**

The truth set of  $P(x)$  is

$$T_P := \{x \in \mathbb{Z}^+ | x \text{ is not divisible by } 3\} = \mathbb{Z}^+ - \{3, 6, 9, 12, 15, 18, 21, 24, 27, 30, 33, 36, \dots\}.$$

The truth set of  $Q(x)$  is

$$T_Q := \{x \in \mathbb{Z}^+ \mid x \text{ is divisible by } 12\} = \{12, 24, 36, 48, 60, 72, \dots\}.$$

(a) **False.**  $T_Q \not\subseteq T_P$ .

A counterexample is when  $x = 36 \in T_Q$  but  $x = 36 \notin T_P$ .

I.e.  $Q(36)$  is true, but  $P(36)$  is not true. Hence,  $Q(x) \Rightarrow P(x)$  is false.

(b) **True.** The truth set of  $\sim Q(x)$  is  $\mathbb{Z}^+ - \{12, 24, 36, \dots\}$ , which contains  $T_P$ . I.e.,  $T_P \subseteq T_{\sim Q}$ . Hence,  $P(x) \Rightarrow \sim Q(x)$  is true.

6. Write a negation for each statement without using *the negation symbol* “ $\sim$ .”

(a)  $\exists z \forall x \forall y, xy = z$

(b)  $\forall x \forall y, (x < 0) \wedge (y \geq 0) \rightarrow (xy \leq 0)$

**Answer:** (a) The negation of  $\exists z \forall x \forall y, xy = z$  can be found step-by-step as follows.

$$\begin{aligned} \sim (\exists z \forall x \forall y, xy = z) &\equiv \forall z \sim (\forall x \forall y, xy = z) \\ &\equiv \forall z \exists x \sim (\forall y, xy = z) \\ &\equiv \forall z \exists x \exists y \sim (xy = z) \\ &\equiv \forall z \exists x \exists y, xy \neq z \end{aligned}$$

(b) The negation of  $\forall x \forall y, (x < 0) \wedge (y \geq 0) \rightarrow (xy \leq 0)$  can be found step-by-step as follows. Consider  $p \wedge q \rightarrow r$ , using the order of operation we have  $(p \wedge q) \rightarrow r$  and its negation is

$$\sim ((p \wedge q) \rightarrow r) \equiv (p \wedge q) \wedge \sim r.$$

From above, that the negation of  $(x < 0) \wedge (y \geq 0) \rightarrow (xy \leq 0)$  is

$$(x < 0) \wedge (y \geq 0) \wedge \sim (xy \leq 0) \equiv (x < 0) \wedge (y \geq 0) \wedge (xy > 0).$$

The negation of  $\forall x \forall y, (x < 0) \wedge (y \geq 0) \rightarrow (xy \leq 0)$  is

$$\begin{aligned} \sim (\forall x \forall y, (x < 0) \wedge (y \geq 0) \rightarrow (xy \leq 0)) &\equiv \exists x \sim (\forall y, (x < 0) \wedge (y \geq 0) \rightarrow (xy \leq 0)) \\ &\equiv \exists x \exists y, \sim ((x < 0) \wedge (y \geq 0) \rightarrow (xy \leq 0)) \\ &\equiv \exists x \exists y, (x < 0) \wedge (y \geq 0) \wedge (xy > 0). \end{aligned}$$

7. Show that each of the following arguments is valid by **universal modus ponens**, **universal modus tollens** and/or **universal transitivity**, or show that it is invalid from the **converse error** or the **inverse error**. In addition, use also the **diagram** to confirm that each argument is valid or invalid.

(a)

“Anyone who has a school email account has a school ID number.”

“Kevin has a school ID number.”

$\therefore$  “Kevin has a school email account.”

(b)

“Anyone who has a school email account has a school ID number.”

“All students have school email accounts.”

“Kim does not have a school ID number.”

∴ “Kim is not a student.”

**Answer:**

(a) “Anyone who has a school email account has a school ID number.”

“Kevin has a school ID number.”

∴ “Kevin has a school email account.”

Let  $P(x)$  be “x has a school email account,”

$Q(x)$  be “x has a school ID number.”

Then we can transform the given argument in the quantified form of **converse error** as follows.

$$\begin{aligned} &\forall x, P(x) \rightarrow Q(x) \\ &Q(a) \text{ for a particular } a = \text{Kevin} \\ &\therefore P(a) \end{aligned}$$

That is, this argument is **invalid** by the converse error.

To use the diagram,

let  $A$  be the set of people who have school email accounts, and

let  $B$  be the set of people who have school ID numbers.

Then

- the first premise tells us that “ $A \subseteq B$ .”
- the second premise tells us that “Kevin is in the set  $B$ ,”
- the conclusion tells us that “Kevin is in the set  $A$ .”

From the diagram, since we only know that “Kevin” is an element in  $B$ , it is possible that “Kevin” is either an element in  $A$  or not an element in  $A$ . That is, the conclusion may not happen because it is possible that “Kevin” does **not** have an email account. Therefore the statement is **invalid**.

(b) “Anyone who has a school email account has a school ID number.”

“All students have school email accounts.”

“Kim does not have a school ID number.”

∴ “Kim is not a student.”

Let  $P(x)$  be “x has a school email account,”

$Q(x)$  be “x has a school ID number,”

$R(x)$  be “x is a student.”

Then we can transform the given argument in the quantified form as follows.

$$\begin{aligned} &\forall x, P(x) \rightarrow Q(x) \\ &\forall x, R(x) \rightarrow P(x) \\ &\sim Q(a) \text{ for a particular } a = \text{Kim} \\ &\therefore \sim R(a) \end{aligned}$$

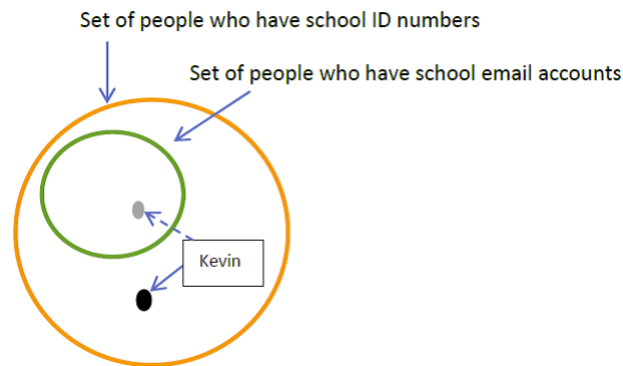


Figure 1: Problem 7 (a)

and using the universal transitivity rule we have

$$\begin{aligned} \forall x, R(x) &\rightarrow P(x) \\ \forall x, P(x) &\rightarrow Q(x) \\ \therefore R(x) &\rightarrow Q(x). \end{aligned}$$

Using the result from the transitivity rule above,  $\forall x, R(x) \rightarrow Q(x)$ , the original argument becomes

$$\begin{aligned} \forall x, R(x) &\rightarrow Q(x) \\ \sim Q(a) &\text{ for a particular } a = \text{Kim} \\ \therefore \sim R(a) \end{aligned}$$

which is valid by universal modus tollens. That is, the original argument is **valid** by **the universal transitivity and universal modus tollens**.

To use the diagram,

let  $A$  be the set of people who have school email accounts,  
let  $B$  be the set of people who have school ID numbers, and  
let  $C$  be the set of students.

Then

the first premise tells us that " $A \subseteq B$ ."

the second premise tells us that " $C \subseteq A$ ."

the third premise tells us that "Kim is **not** in the set  $B$ ,"

the conclusion tells us that "Kim is **not** in the set  $C$ ."

That is, the second and the third premise gives  $C \subseteq A \subseteq B$ , which implies that  $C \subseteq B$ . From the diagram, since "Kim" is not an element in  $B$ , it is **impossible** that "Kim" is an element in  $C$ , which is inside  $B$ . I.e., the conclusion that "Kim is not an element in  $C$ " is **always** true. Hence, the given argument is **valid**.

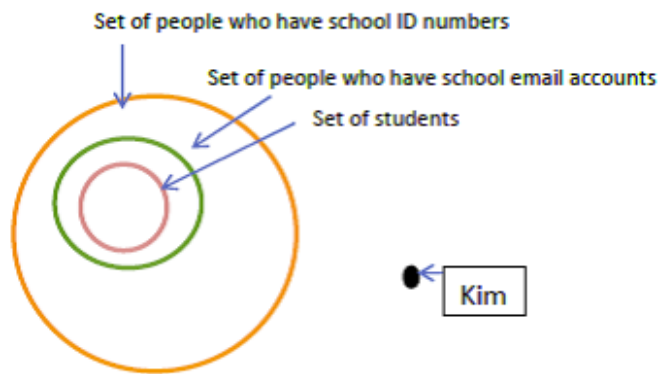


Figure 2: Problem 7 (b)