Limits and Continuity

Consider the rational function $f(x)=\dfrac{x^2-1}{x-1}$. Clearly we cannot use $x = 1$.

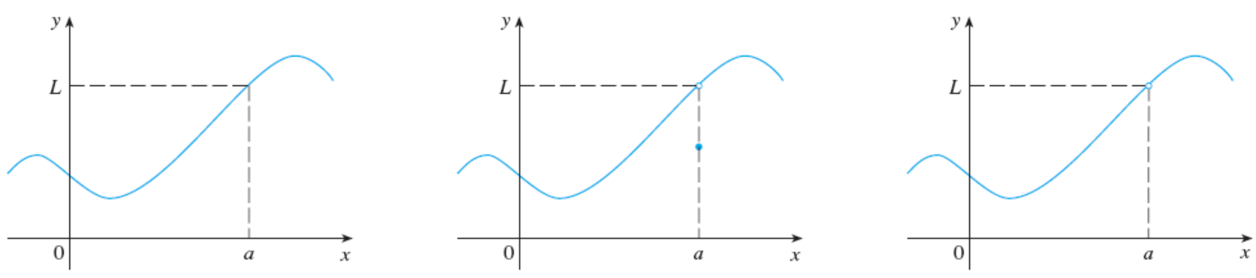
We want to know what happens if our x values approach $x = 1$. In this case

$x < 1$	$f(x)$	$x > 1$	$f(x)$
0.9	1.9	1.01	2.01
0.99	1.99	1.001	2.001
0.999	1.999	1.0001	2.0001

From the tables, we can see that if our x values approach $x = 1$, then our $f(x)$ (or y) values approach 2. This idea is expressed using the limit. We write $\lim_{x\to 1} f(x)=2$ and we read "The limit of $f(x)$ as x approaches one is two".

In general,

$$\lim_{x\to a} f(x)=L$$



(a) (b) (c)

In all of these cases above, $\lim_{x\to a} f(x)=L$.

One sided Limits

Consider the split definition function

$$f(x)=\begin{cases} x^2-1 & if \quad x\ge 0 \\ \dfrac{1}{2}x+2 & if \quad x<0 \end{cases}$$

We have $\lim_{x\to 0} f(x)$ does not exist.

However,

$$\lim_{x\to 0^+} f(x)=-1 \quad \text{and} \quad \lim_{x\to 0^-} f(x)=2\,.$$

We write

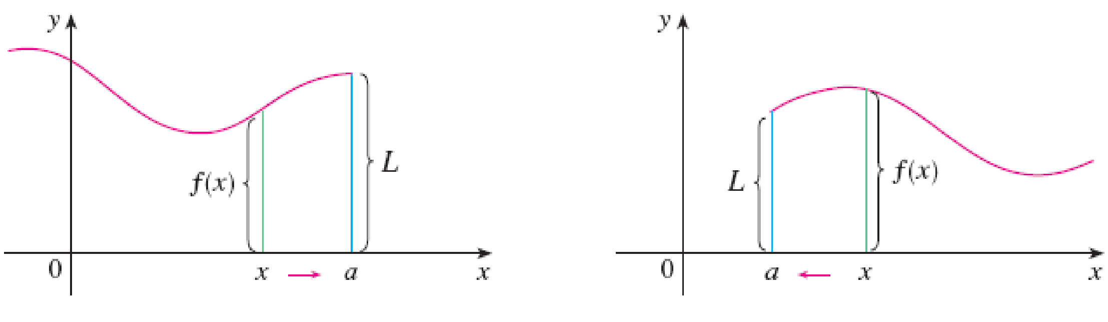
$$\lim_{x\to a^-} f(x)=L$$

"The left – hand limit of $f(x)$ as x approaches a (OR the limit of $f(x)$ as x approaches a from the left) is equal to L."

Similarly, we write

$$\lim_{x\to a^+} f(x)=L$$

"The right – hand limit of $f(x)$ as x approaches a (OR the limit of $f(x)$ as x approaches a from the right) is equal to L."



(a) $\lim_{x\to a^-} f(x)=L$ (b) $\lim_{x\to a^+} f(x)=L$

▪ Computing Limits

Limit Laws : Suppose that c is a constant and the limits

$$\lim_{x\to a} f(x) \quad \text{and} \quad \lim_{x\to a} g(x)$$

exist. Then

1. $\lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$

2. $\lim_{x\to a}[f(x)-g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$

3. $\lim_{x\to a}[cf(x)] = c\lim_{x\to a} f(x)$

4. $\lim_{x\to a}[f(x)g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$

5. $\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ where $\lim_{x\to a} g(x) \neq 0$

6. $\lim_{x\to a}[f(x)]^n = [\lim_{x\to a} f(x)]^n$ where n is a positive integer

7. $\lim_{x\to a} c = c$

8. $\lim_{x\to a} x = a$

9. $\lim_{x\to a} x^n = a^n$ where n is a positive integer

10. $\lim_{x\to a}\sqrt[n]{x} = \sqrt[n]{a}$ If n is even, we assume that $a > 0$

11. $\lim_{x\to a}\sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$ If n is even, we assume that $\lim_{x\to a} f(x) > 0$

Direct Substitution Property: If f is a polynomial or rational function and a is in the domain of f, then

$$\lim_{x\to a} f(x) = f(a)$$

Example: Find the following limits.

$$(a)\lim_{x\to 1}\frac{3x+\sqrt{x+3}}{x-2}$$

$$(b)\lim_{x\to 2}\frac{2x-8+x^2}{x-2}$$

$$(c)\lim_{x\to -1} x+5-\frac{(x+1)^2}{x^2-x-2}$$

$$(d)\lim_{x\to 2}\frac{\sqrt{x-1}-1}{x-2}$$

$$(e)\lim_{x\to 0}\frac{|x|}{x}$$

$$(f)\lim_{x\to 3} f(x) \quad \text{when} \quad f(x)=\begin{cases}\sqrt{x+6} & if\ \ x>3\\ x^2 & if\ \ x<3\end{cases}$$

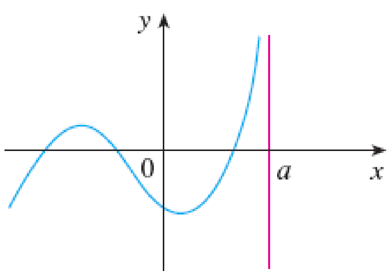
- **Infinite Limits**

Vertical Asymptote

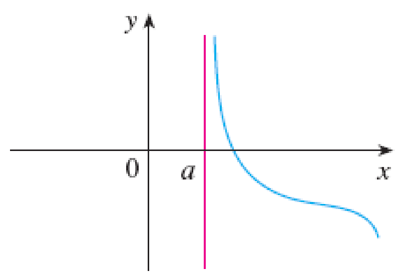
Definition: The line $x = a$ is called a *vertical asymptote* of the curve $y = f(x)$ if at least one of the following statements is true:

$$\lim_{x\to a} f(x) = \infty \qquad \lim_{x\to a^-} f(x) = \infty \qquad \lim_{x\to a^+} f(x) = \infty$$
$$\lim_{x\to a} f(x) = -\infty \qquad \lim_{x\to a^-} f(x) = -\infty \qquad \lim_{x\to a^+} f(x) = -\infty$$

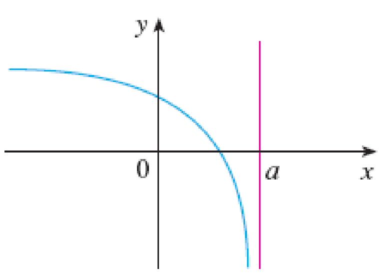
For example,



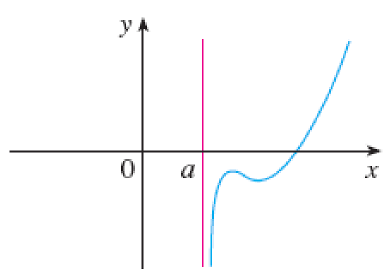
(a) $\lim_{x\to a^-} f(x) = \infty$



(b) $\lim_{x\to a^+} f(x) = \infty$



(c) $\lim_{x\to a^-} f(x) = -\infty$



(d) $\lim_{x\to a^+} f(x) = -\infty$

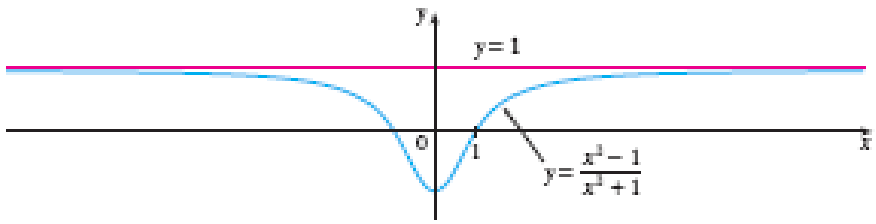
Horizontal Asymptote

Definition: The line $y = L$ is called a *horizontal asymptote* of the curve $y = f(x)$ if either

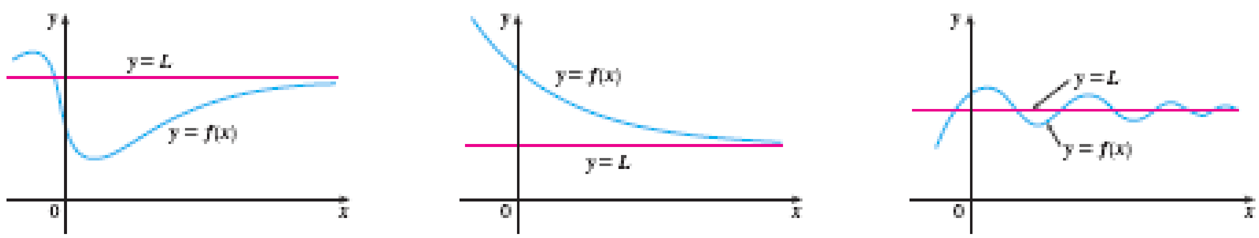
$$\lim_{x\to\infty} f(x) = L \qquad \lim_{x\to-\infty} f(x) = L$$

Example: Consider $f(x) = \dfrac{x^2-1}{x^2+1}$

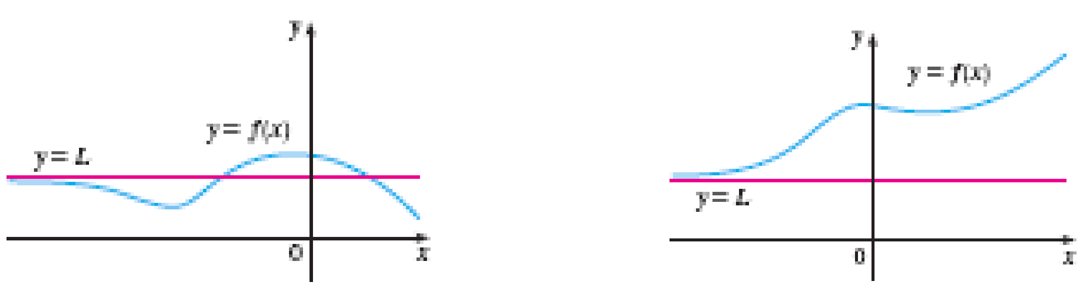
x	0	± 1	± 2	± 3	± 4	± 10	± 50	± 100	± 1000
$f(x)$	-1	0	0.6	0.8	0.882	0.98	0.9992	0.9998	0.999998



Examples illustrating $\lim_{x\to\infty} f(x) = L$:



Examples illustrating $\lim_{x\to-\infty} f(x) = L$:



Theorem: If $r > 0$ is a rational number, then

$$\lim_{x\to\infty} \frac{1}{x^r} = 0$$

If $r > 0$ is a rational number such that x^r is defined for all x, then

$$\lim_{x\to-\infty} \frac{1}{x^r} = 0$$

Example: Find the horizontal asymptotes and vertical asymptotes of the graph of the function.

(1) $f(x) = \dfrac{x^2 - x}{x^2 - 1}$

(2) $f(x) = \dfrac{x^2 - x}{x - 1}$

Infinite Limits at Infinity

The notation

$$\lim_{x\to\infty} f(x) = \infty$$

is used to indicate that the values of f(x) become large. Similar meaning are attached to

$$\lim_{x\to\infty} f(x) = -\infty \quad \lim_{x\to-\infty} f(x) = \infty \quad \lim_{x\to-\infty} f(x) = -\infty$$

Example: Find

(1) $\lim_{x\to\infty} 2x^3$

(2) $\lim_{x\to-\infty} 2x^3$

(3) $\lim_{x\to-\infty} 3x^2$

(4) $\lim_{x\to\infty} \dfrac{x^2 - x}{x-1}$

(5) $\lim_{x\to-\infty} \dfrac{x^2 + x}{x+3}$

(6) $\lim_{x\to\infty} \dfrac{\sqrt{9x^6 - x}}{x^3 + 1}$

(7) $\lim_{x\to-\infty} \left(x + \sqrt{x^2 + 2x}\right)$

(8) $\lim_{x\to-\infty} \dfrac{x}{\sqrt{x^2 + 4}}$

(9) $\lim_{x\to-\infty} \left(x^4 + x^5\right)$

▪ Continuity

Notice that the limit of a function as *x* approaches *a* can often be found simply by calculating the value of the function at *a*. Functions with this property are called continuous at *a*.

Definition: A function is said to be continuous at *a* if

(1) $f(a)$ is defined (that is, a is in the domain of f).

(2) $\lim_{x\to a} f(x)$ exists.

(3) $\lim_{x\to a} f(x) = f(a)$.

Example: Given

$$f(x)=\begin{cases}\dfrac{x^2-1}{x-1} & if\ \ x\neq 1\\ 2 & if\ \ x=1\end{cases}$$

Is *f* continuous at 1?

Example: Where are each of the following functions discontinuous?

$(a)\ f(x)=\dfrac{x^2+x-2}{x-1}$

$(b)\ f(x)=\begin{cases}\dfrac{x^2+x-2}{x-1} & if\ \ x\neq 1\\ 2 & if\ \ x=1\end{cases}$

Theorem: If *f* and g are continuous at *a* and *c* is *a* constant, then the following functions are also continuous at *a*:

1. $f + g$
2. $f - g$
3. cf
4. fg
5. f / g if $g(a) \neq 0$

Theorem: The following types of functions are continuous at every number in their domains:

- Polynomials
- Root functions
- Rational functions
- Trigonometric functions

Example: Given

$$f(x) = \begin{cases} 1 + x^2 & if \ x \le 0 \\ 2 - x & if \ 0 < x \le 2 \\ (x-2)^2 & if \ x > 2 \end{cases}$$

Find the numbers at which *f* is discontinuous.